E-MODULE
ON
CRISP AND FUZZY RELATIONS
FOR
CLASS:-MSC(IT)II-SEM
SUBJECT:-FUZZY SYSTEMS

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CRISP RELATION

- **Definition (Product set):**

Let A and B be two nonempty sets, the product set or Cartesian product $A \times B$ is defined as follows,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

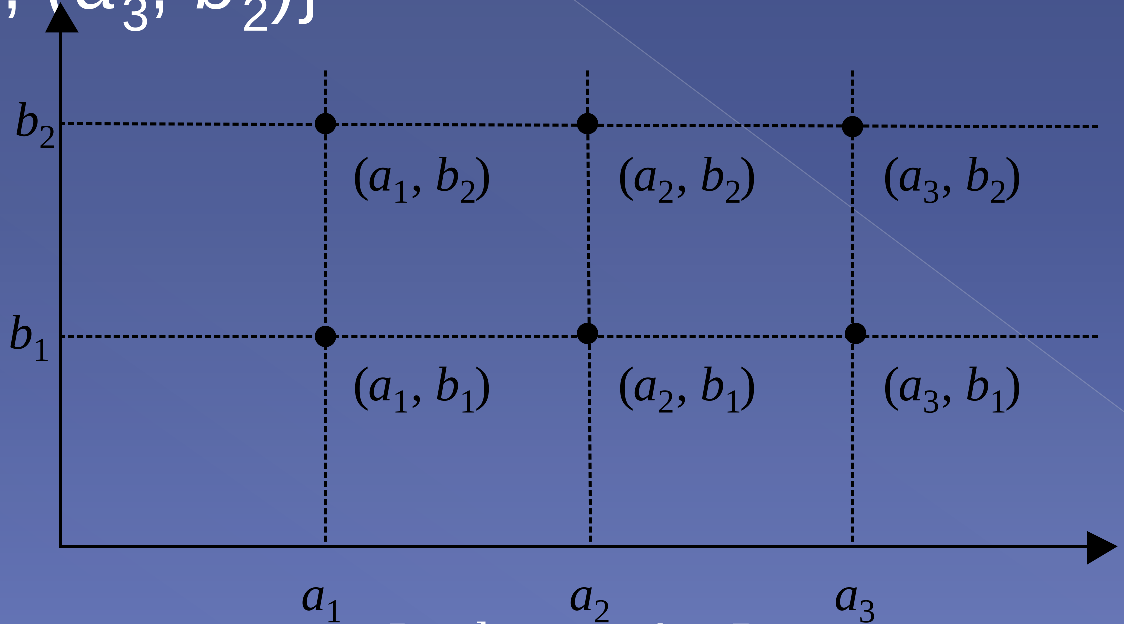
- Extension to n sets

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

EXAMPLE OF CRISP RELATION

Example: $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$

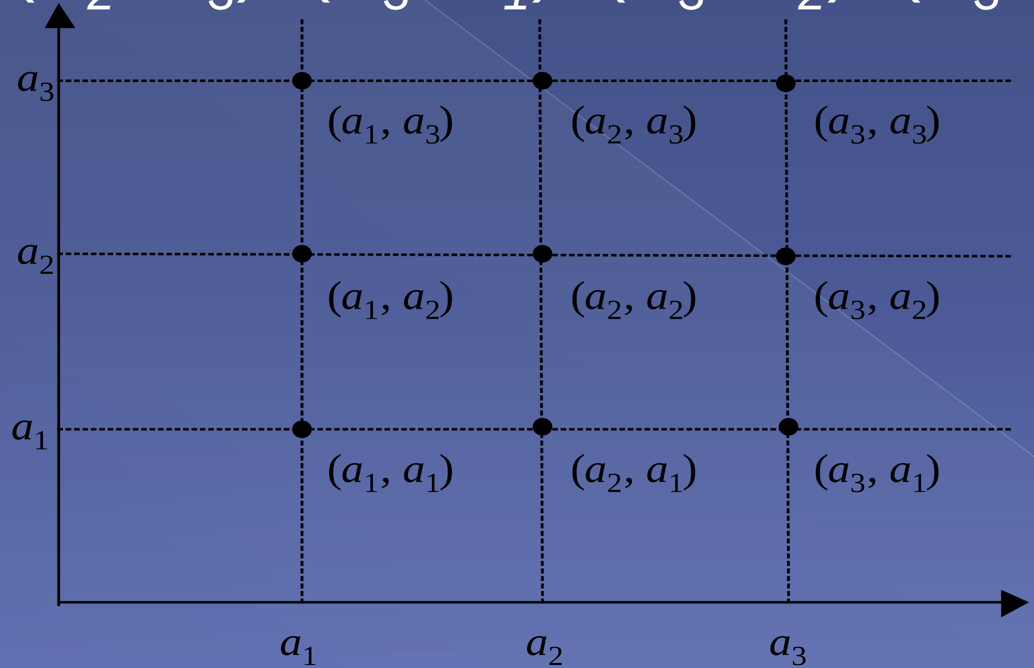
$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$



Product set $A \times B$

CARTESION PRODUCT ON SAME RELATION

$A \times A = \{(a_1, a_1), (a_1, a_2), (a_1, a_3), (a_2, a_1), (a_2, a_2), (a_2, a_3), (a_3, a_1), (a_3, a_2), (a_3, a_3)\}$



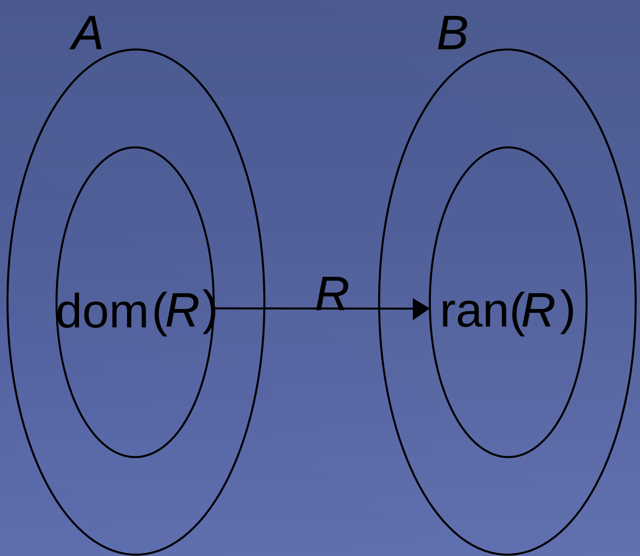
Cartesian product $A \times A$

DOMAIN AND RANGE OF CRISP RELATION

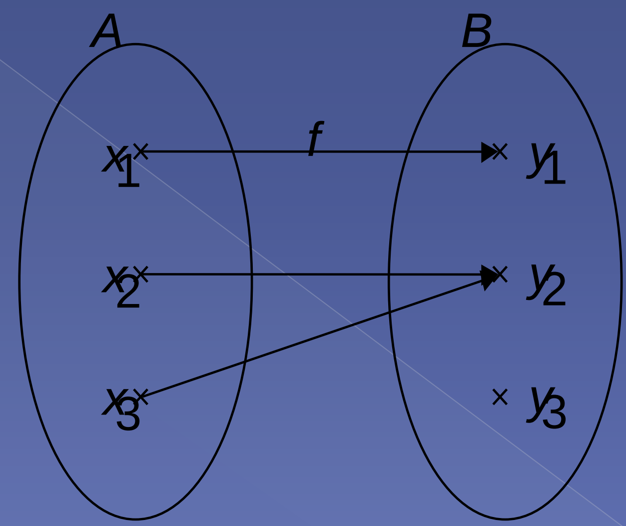
- Domain and Range

dom(R) = { x | $x \in A$, $(x, y) \in R$ for some $y \in B$ }

ran(R) = { y | $y \in B$, $(x, y) \in R$ for some $x \in A$ }



dom(R) , ran(R)



Mapping $y = f(x)$

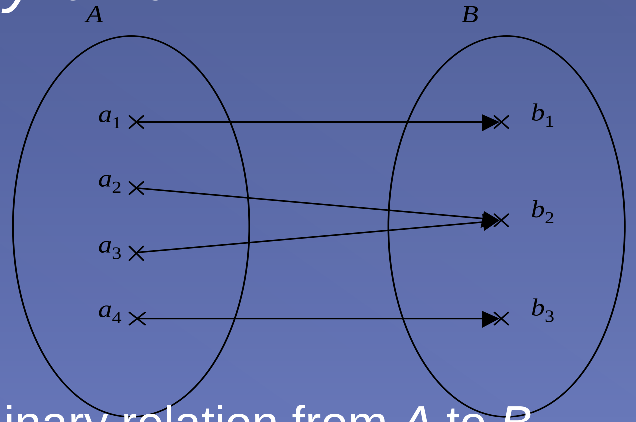
REPRESENTATION OF CRISP RELATIONS

(1)Bipartigraph

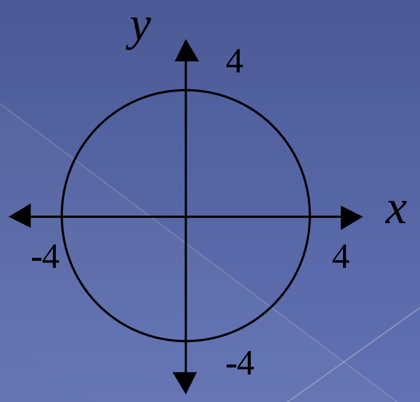
representing the relation by drawing arcs or edges

(2)Coordinate diagram

plotting members of *A* on *x* axis and that of B on *y* axis



Binary relation from *A* to *B*



Relation of $x^2 + y^2 = 4$

(3) Matrix

$M_R = (m_{ij})$

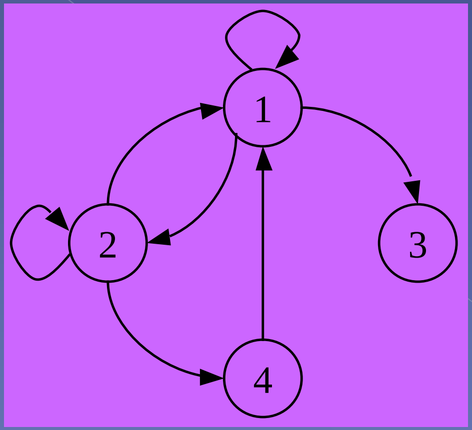
$i = 1, 2, 3, \ldots, m$
$j = 1, 2, 3, \ldots, n$

(4) Digraph

$$m_{ij} = \begin{cases} 1, & (a_i, b_j) \in R \\ 0, & (a_i, b_j) \notin R \end{cases}$$

R	b_1	b_2	b_3
a_1	1	0	0
a_2	0	1	0
a_3	0	1	0
a_4	0	0	1

Matrix



Directed graph

OPERATIONS ON CRISP RELATIONS

- **Operations on relations $R, S \subseteq A \times B$**

  (1) Union $T = R \cup S$

  If $(x, y) \in R$ or $(x, y) \in S$, then $(x, y) \in T$

  (2) Intersection $T = R \cap S$

  If $(x, y) \in R$ and $(x, y) \in S$, then $(x, y) \in T$.

  (3) Complement

  If $(x, y) \notin R$, then $(x, y) \in R^C$ $\overline{R}$

  (4) Inverse

  $R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R, x \in A, y \in B\}$

  (5) Composition T

  $R \subseteq A \times B, S \subseteq B \times C, T = S \bullet R \subseteq A \times C$

  $T = \{(x, z) \mid x \in A, y \in B, z \in C, (x, y) \in R, (y, z) \in S\}$

TYPES OF RELATION ON A SET

- Reflexive relation

  $x \in A \to (x, x) \in R$ or $\mu_R(x, x) = 1, \forall x \in A$

  - irreflexive

    if it is not satisfied for some $x \in A$
  - antireflexive

    if it is not satisfied for all $x \in A$
- Symmetric relation

  $(x, y) \in R \to (y, x) \in R$ or $\mu_R(x, y) = \mu_R(y, x), \forall x, y \in A$

  - asymmetric or nonsymmetric

    when for some $x, y \in A$, $(x, y) \in R$ and $(y, x) \notin R$.
  - antisymmetric

    if for all $x, y \in A$, $(x, y) \in R$ and $(y, x) \notin R$

INTRODUCTION TO FUZZY RELATIONS

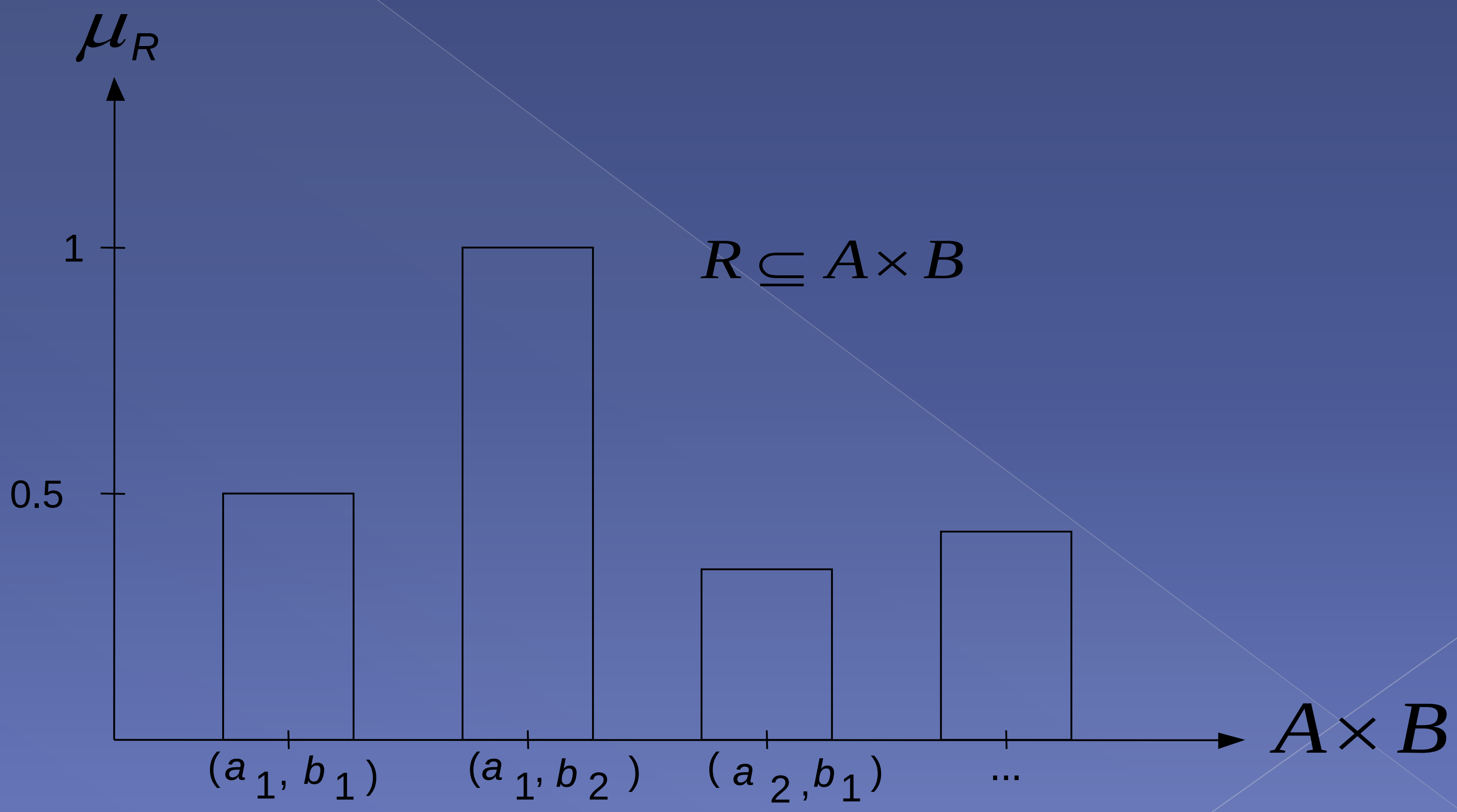
- Definition of fuzzy relation
  - Crisp relation
    - membership function $\mu_R(x, y)$

$\mu_R : A \times B \to \{0, 1\}$

$$\mu_R(x, y) = \begin{cases} 1 \text{ iff } (x, y) \in R \\ 0 \text{ iff } (x, y) \notin R \end{cases}$$

  - **Fuzzy relation**
    - $\mu_R : A \times B \to [0, 1]$
    - $R = \{((x, y), \mu_R(x, y)) \mid \mu_R(x, y) \geq 0, \; x \in A, \; y \in B\}$

Fuzzy Relation



Fuzzy relation as a fuzzy set

EXAMPLE OF FUZZY RELATION

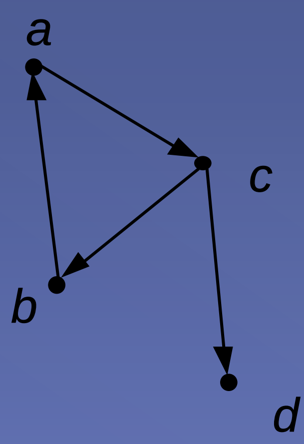
- **Example**

  <u>Crisp relation *R*</u>

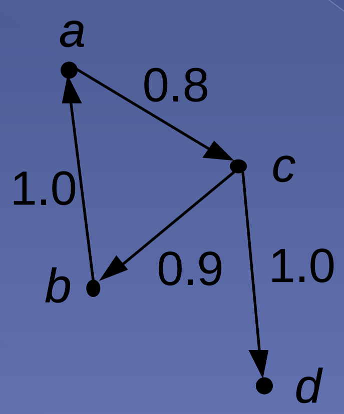
  $\mu_R(a, c) = 1$, $\mu_R(b, a) = 1$, $\mu_R(c, b) = 1$ and $\mu_R(c, d) = 1$.

  <u>Fuzzy relation R</u>

  $\mu_R(a, c) = 0.8$, $\mu_R(b, a) = 1.0$, $\mu_R(c, b) = 0.9$, $\mu_R(c, d) = 1.0$



(a) Crisp relation



(b) Fuzzy relation

A \ A	*a*	*b*	*c*	*d*
a	0.0	0.0	0.8	0.0
b	1.0	0.0	0.0	0.0
c	0.0	0.9	0.0	1.0
d	0.0	0.0	0.0	0.0

corresponding matrix

crisp and fuzzy relations

OPERATIONS ON FUZZY RELATIONS

1) **Union relation**

$\forall (x, y) \in A \times B$

$\mu_{R \cup S}(x, y) = \text{Max}[\mu_R(x, y), \mu_S(x, y)] = \mu_R(x, y) \vee \mu_S(x, y)$

2) **Intersection relation**

$\mu_{R \cap S}(x) = Min[\mu_R(x, y), \mu_S(x, y)] = \mu_R(x, y) \wedge \mu_S(x, y)$

3) **Complement relation**

$\forall (x, y) \in A \times B$

$\mu_{\bar{R}}(x, y) = 1 - \mu_R(x, y)$

4) **Inverse relation**

For all $(x, y) \subseteq A \times B$, $\mu_{R^{-1}}(y, x) = \mu_R(x, y)$

FUZZY OPERATIONS EXAMPLE

- Examples

M_R	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.0	1.0	0.0

M_S	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.6	0.9	0.3

$M_{R \cup S}$	a	b	c
1	0.3	0.2	1.0
2	0.8	1.0	1.0
3	0.6	1.0	0.3

$M_{R \cap S}$	a	b	c
1	0.3	0.0	0.1
2	0.1	0.8	1.0
3	0.0	0.9	0.0

$M_{\bar{R}}$	a	b	c
1	0.7	0.8	0.0
2	0.2	0.0	0.0
3	1.0	0.0	1.0

Types of Fuzzy Relations

- Reflexive
  - Irreflexive
  - Antireflexive
  - Epsilon Reflexive

$R(x,x) = 1 \text{ for all } x \in X$

$R(x,x) \neq 1 \text{ for some } x \in X$

$R(x,x) \neq 1 \text{ for all } x \in X$

$R(x,x) \geq \varepsilon \text{ for all } x \in X$

- Symmetric
  - Asymmetric
  - Antisymmetric

$R(x,y) = R(y,x) \text{ for all } x \in X$

$R(x,y) \neq R(y,x) \text{ for some } x \in X$

$R(x,y) > 0 \text{ and } R(y,x) > 0 \rightarrow x = y \text{ for all } x, y \in X$

Types of Fuzzy Relations

- Transitive (max-min transitive)

$$R(x,z) \geq \max_{y \in Y} \min[R(x,y), R(y,z)] \text{ for all } x,z \in X$$

  - Non-transitive:
    For some (x,z), the above do not satisfy.
  - Antitransitive:

$$R(x,z) < \max_{y \in Y} \min[R(x,y), R(y,z)] \text{ for all } x,z \in X$$

- Example: X = Set of cities, R="very far"
  Reflexive, symmetric, non-transitive

PROPERTIES OF FUZZY RELATIONS

A fuzzy relation R(X,X) is called

- reflexive if and only if ∀x ∈ X : R(x, x) = 1,
- symmetric if and only if ∀x, y ∈ X : R(x, y) = R(y, x),
- transitive if it satisfies R(x, z) ≥ max y∈Y min{R(x, y), R(y, z)}, ∀(x, z) ∈ X 2 .

R =

	y_1	y_2	y_3	y_4	y_5
x_1	.9	1	0	0	0
x_2	0	.4	0	0	0
x_3	0	.5	1	.2	0
x_4	0	0	0	1	.4
x_5	0	0	0	0	.5
x_6	0	0	0	0	.2

$$\mu_{dom(R)}(x_1) = 1.0$$

$$\mu_{dom(R)}(x_2) = 0.4$$

$$\mu_{dom(R)}(x_3) = 1.0$$

$$\mu_{dom(R)}(x_4) = 1.0$$

$$\mu_{dom(R)}(x_5) = 0.5$$

$$\mu_{dom(R)}(x_6) = 0.2$$

COMPOSITION OF FUZZY RELATIONS

- Max-min composition

$$\forall (x, y) \in A \times B,\ \forall (y, z) \in B \times C$$

$$\mu_{S \circ R}(x,\ z) = \max_{y}[\min(\mu_R(x,\ y),\ \mu_S(y,\ z))]$$

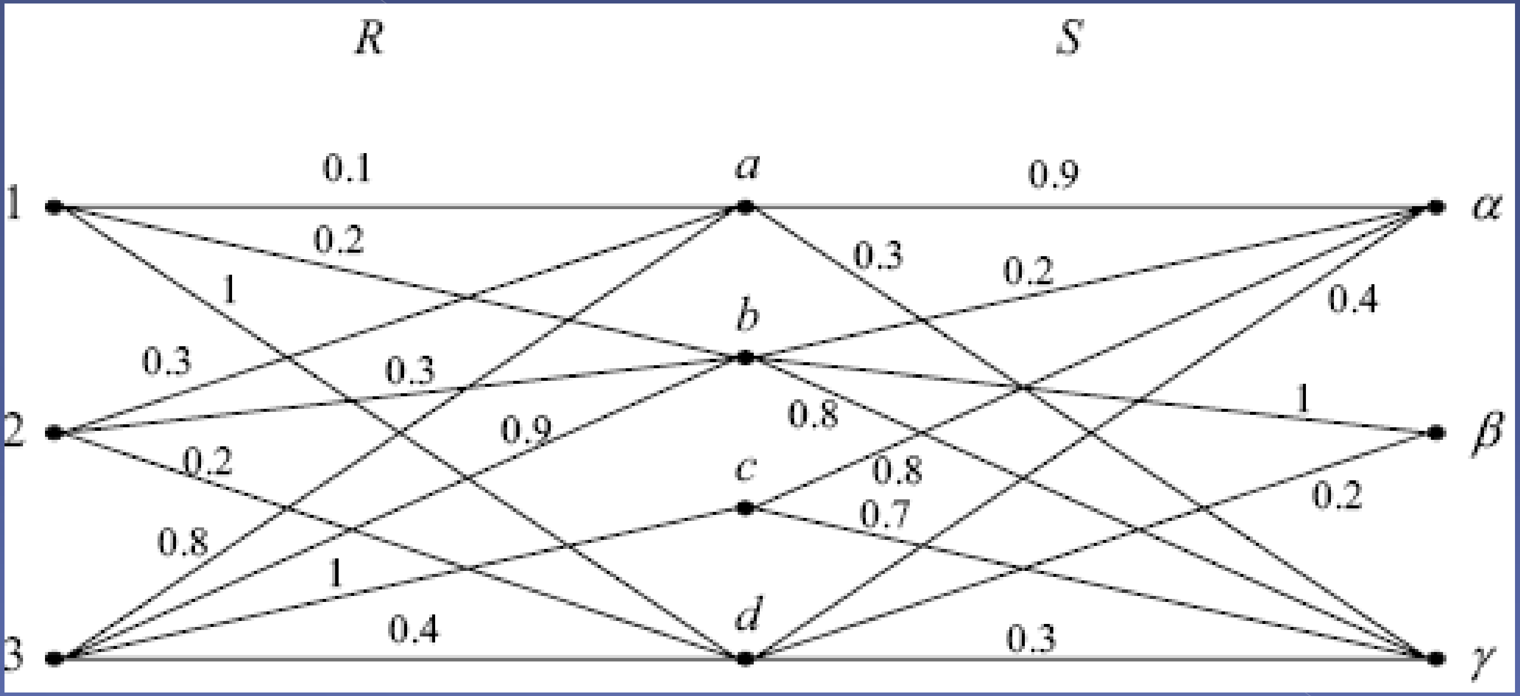
$$= \vee_{y}[\mu_R(x,\ y) \wedge \mu_S(y,\ z)]$$

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

COMPOSITION OF FUZZY RELATIONS



COMPOSITION OF FUZZY RELATIONS

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

$$\mu_{S \circ R}(1,\ \alpha) = \max[\min(0.1, 0.9), \min(0.2, 0.2), \min(0.0, 0.8), \min(1.0, 0.4)]$$

$$= \max[0.1, 0.2, 0.0, 0.4] = 0.4$$

COMPOSITION OF FUZZY RELATIONS

- Example

R	a	b	c	d
1	0.1	0.2	0.0	1.0
2	0.3	0.3	0.0	0.2
3	0.8	0.9	1.0	0.4

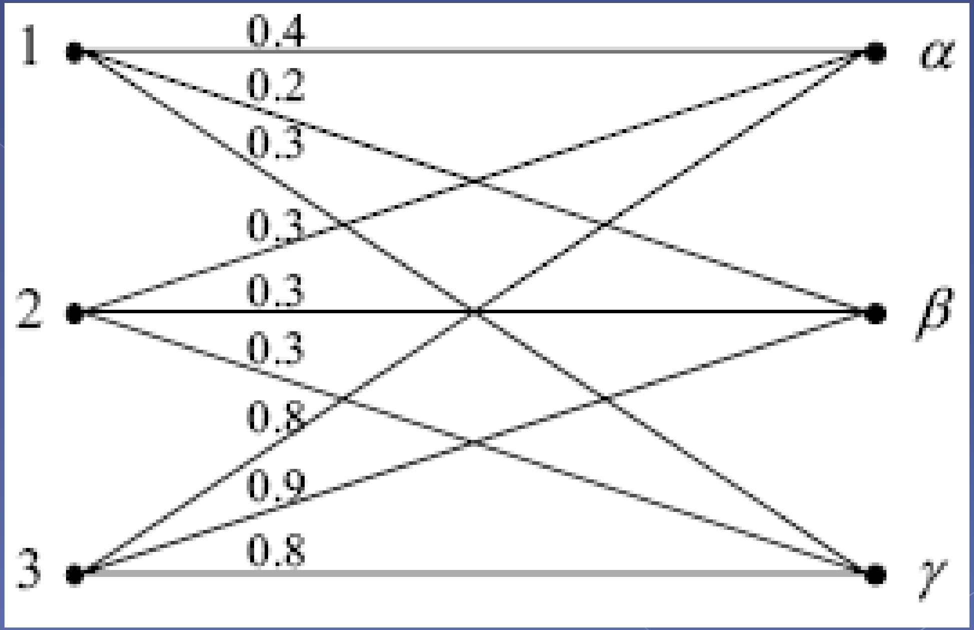
S	α	β	γ
a	0.9	0.0	0.3
b	0.2	1.0	0.8
c	0.8	0.0	0.7
d	0.4	0.2	0.3

$$\mu_{S \bullet R}(1, \beta) = \max[\min(0.1, 0.0), \min(0.2, 1.0), \min(0.0, 0.0), \min(1.0, 0.2)]$$
$$= \max[0.0, 0.2, 0.0, 0.2] = 0.2$$

COMPOSITION OF FUZZY RELATIONS

S•R	α	β	γ
1	0.4	0.2	0.3
2	0.3	0.3	0.3
3	0.8	0.9	0.8

S•R



THANK YOU