FUZZY LOGIC: BASIC CONCEPTS, OPERATIONS & PROPERTIES

Urvashi Mishra
Asst. Prof. in Comp. Sc.
Hans Raj Mahila Maha Vidyalaya, Jalandhar

Definition

- Narrow sense-
  - “Fuzzy logic refers to the logical system that generalizes classical two-valued logic for reasoning under uncertainty”

- Broad Sense-
  - Fuzzy logic refers to all of the theories and technologies that employ fuzzy sets, that are classes with unsharp boundaries”

Fuzzy Logic

- Does not have a well defined sharp boundary.
- Reasoning under uncertainty.
- Membership becomes a matter of degree
- Ease of describing human knowledge involving vague concepts.
- Cost-effective solution to real-world problems.

Applications

- Fuzzy Control
- Fuzzy Pattern Recognition
- Fuzzy Arithmetic
- Fuzzy Mathematical Programming
- Fuzzy Probability Theory
- Fuzzy Decision Analysis
- Fuzzy Neural Networks Theory
- Fuzzy Topology
- Fuzzy Generic Algorithm
- Fuzzy Artificial Intelligence

Fuzzy Logic for Control

- Motivations
  - To deal with complex systems
  - The ease of describing human knowledge
- Applications of Fuzzy Logic Control
  - Consumer Products: Cameras, Camcorders, Washing Machines, Refrigerators
  - Automotive and Power Generation: Power Train and transmission control, engine control
  - Industrial Process Control: Refining, Distillation, Chemical Processes, Cement Kiln
  - Robotics and Manufacturing: Electrical Discharge Machine

Basic features

- Fuzzy Sets - *Sets with smooth boundaries*
- Linguistic Variables – *variables whose values are both qualitatively and quantitatively described by a fuzzy set*
- Possibility Distribution – *constraints on the value of a linguistic imposed by assigning it a fuzzy set*
- Fuzzy-if-then rules – *a knowledge representation scheme for describing a functional mapping or a logic formula that generalizes an implication in two-valued logic.*

I. Fuzzy Sets

- A set with smooth boundary
- Generalizes classical set theory to allow partial membership
- Smooth and gradual transition from regions outside the set to those in the set
- Representation of a set by membership function
- Easy to express knowledge using linguistic terms
- Knowledge expressed using linguistic terms is comprehensible

II. Linguistic variables

- Enables its values to be described both qualitatively and quantitatively.
  - Qualitatively by using linguistic term i.e. a symbol for the name of the fuzzy set
  - Quantitatively by membership function expressing the meaning of the fuzzy set
- Used to express concepts and knowledge in human communication.
- For example:
  - TradingQuantity is Heavy

LINGUISTIC VARIABLE= SYMBOLIC VARIABLE+NUMERIC VARIABLE

- Symbolic Variable:
  - A variable whose value is a symbol
  - Used in artificial intelligence, decision sciences

- Numeric Variable:
  - A variable whose value is a number
  - Used in science, engineering, mathematics, medicine, etc

III. Possibility Distribution

- Generalizes binary distinction between possible vs. impossible to a matter of degree
- ∏ denotes a possibility distribution.
- Possibility distribution of X defined by A's membership function

  $\Pi_X(x) = \mu_A(x)$

- PD states the degree of ease (i.e. possibility) for the variable to take a certain value without indicating the likelihood that the variable has such a

IV. Fuzzy If-Then Rules

- Generalization of a logic inference called modus ponens.
- It is a knowledge representation scheme for capturing knowledge that is imprecise and inexact by nature
- Perform inference under partial matching.
- Computes the degree the input data matches the condition of a rule.
- Structure:

IF <antecedent> THEN <consequent>

The antecedent describes a condition and consequent describes a conclusion.

Fuzzy Sets vs. Classical Sets

Classical Set	Fuzzy Set
Collection of objects in a given domain.	Generalization of classical set
Object either belongs to the set or does not belong to the set.	Object partially belongs to the set.
Sharp boundary between members of the set and those not in the set	Smooth boundary between members and non-members of the set.
Example: set of married people	Example: set of happily married people

Set theory

- A set can be defined in two ways:
  - Extensional definition: By enumerating its elements e.g. A= {UT, TAMU} represents the set of universities of Texas having student population greater than 35000.
  - Intensional definition: By describing the common properties of its elements e.g.

A= {u|tot-stud-pop(u) > 35000 AND state(u) = Texas}

  - S1 is subset of S2 if every element of S1 is present in S2 (S1⊂S2)
  - Inverse of subset is superset referred to as inclusion. (S2⊃S1)

Set Operations

- **Union**
  - $A \cup B = \{x | x \in A \text{ or } x \in B\}$
- **Intersection**
  - $A \cap B = \{x | x \in A \text{ and } x \in B\}$
- **Complement**
  - $A^C = \{x \mid x \in U \text{ and } x \notin A\}$, where U is the universe of discourse
- **Difference**
  - $A \setminus B = \{x | x \in A \text{ and } x \notin B\}$
  - Thus $A^C = U \setminus A$

- Union
- Intersection
- Complement
  - , where U is the universe of discourse
- Difference
  - B}

Thus =U\A

Fundamental Properties of Set

- Commutative Law
- Commutative Law
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- Associative law
- Associative law
  - $A \cap (B \cap C) = (A \cap B) \cap C$
  - $A \cup (B \cup C) = (A \cup B) \cup C$
- Distributive Law
- Distributive Law
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Law of Double Complementation
- Law of Double Complementation
  - $(A')' = A$

Set Properties Contd.

- DeMorgan’s Law
  - $(A \cup B)' = A' \cap B'$
  - $(A \cap B)' = A' \cup B'$
- Law of Excluded Middle
  - $A \cup A' = U$
- Law of Contradiction
  - $A \cap A' = \phi$
- Law of Tautology
  - $A \cup A = A \, ; A \cap A = A$
- Law of Absorption
  - $A \cap (A \cup B) = A$
  - $A \cup (A \cap B) = A$

Principle of Duality

- For any law described in previous slides, if we replace
  - each union with intersection,
  - each intersection with union
  - $\emptyset$ with U
  - U with $\emptyset$

the resulting equation is another valid law in Boolean algebra e.g. the law of excluded middle and law of contradiction are dual laws.

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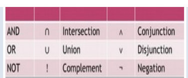
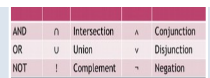
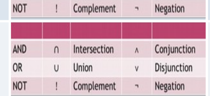
Representation of Fuzzy Sets

- Fuzzy sets allow partial membership and membership in fuzzy sets becomes a matter of degree which is a number between 0 and 1 denoted by Greek symbol μ.
- Fuzzy set can be defined in two ways:
  - By enumerating membership values of elements in the set (completely or partially)
  - By defining the membership function mathematically
- Fuzzy set defined through enumeration
  - $A = \sum \mu_A(x_i)/x_i$
    - Summation and addition operators refer to the union (disjunction) operation

Operations of Fuzzy Sets

- Basic Notations

AND		Intersection		Conjunction
OR		Union		Disjunction
NOT		Complemen t	¬	Negation

- Union and Intersection of Fuzzy Sets
- Complement of a Fuzzy Set
- Subsethood

I. Union and Intersection of Fuzzy Sets

- Consider two logic statements p & q
- Consider two logic statements p & q
  - $p \text{ is true } iff\ x \in A$
  - $q \text{ is true } iff\ x \in B$
  - $(p \wedge q) \text{ is true } iff\ x \in A \cap B$
  - $(p \vee q) \text{ is true } iff\ x \in A \cup B$
  - $\neg p \text{ is true } iff\ x \in A'$
- Fuzzy Notations
  - Fuzzy Disjunction $\mu_{AUB}(x)=max\{\mu_A(x),\mu_B(xi)\}$
  - Fuzzy Conjunction $\mu_{A\cap B}(x)=min\{\mu_A(x),\mu_B(xi)\}$
  - Fuzzy Complement $\mu_{A'}(x)=1-\mu_A(x)$

- Fuzzy Notations
  - Fuzzy Disjunction
  - Fuzzy Conjunction }
  - Fuzzy Complement =1-

- One choice for fuzzy conjunction and fuzzy disjunction is using min and max
- Another common pair for fuzzy disjunction and fuzzy conjunction is algebraic product (for conjunction) and algebraic sum (for disjunction.
  - $\mu_{A \cap B}(x) = \mu_A(x) \times \mu_B(x)$
  - $\mu_{A \cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \times \mu_B(x)$

II. Complement of a Fuzzy Set

- Complement reflects negation
- Let A be a fuzzy set defined over U, then its complement ¬A is defined in terms of , as =*1-*
- The point where = , is called Break even point
- Break even point of a fuzzy set A always ahs 0.5 membership degree in A.

- Complement reflects negation
- Let A be a fuzzy set defined over U, then its complement ¬A is defined in terms of $\mu_A(x)$, as $\mu_{\neg A}(x) = 1 - \mu_A(x)$
- The point where $\mu_{\neg A}(x) = \mu_A(x)$, is called Break even point
- Break even point of a fuzzy set A always ahs 0.5 membership degree in A.

III. Subsethood

- A fuzzy set A in the universe U is a subset of another fuzzy set B, if for every element x in U, its membership degree in A is less or equal to its membership degree in B.

i.e. $A \subseteq B \iff \forall x \in U \; \mu_A(x) \le \mu_B(x)$

- If A is a subset of B, then membership in A implies membership in B i.e.

$A \subseteq B \iff (\forall x \in U \; x \in A \rightarrow x \in B)$

- A fuzzy set A in the universe U is a subset of another fuzzy set B, if for every element x in U, its membership degree in A is less or equal to its membership degree in B.

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- Two types of approaches to define fuzzy subsethood:
  - Subsethood based on logic

    $$s(A,B) = \inf[\max((1 - \mu_A(x)), \mu_B(x))]$$

)]
  - Subsethood based on probabilistic foundation

    $$s(A,B) = \frac{\sum min(\mu_A(x), \mu_B(x))}{\sum \mu_A(X)}$$

Properties of Fuzzy Sets

- Cardinality of Fuzzy Sets
- Height: normal versus Subnormal
- Support and Alpha-level cuts
- Resolution Identity

I. Cardinality of Fuzzy Sets

- Cardinality is total number of elements in the list but in case of fuzzy it is the summation of all the membership degrees

$$Card(A) = \sum_{xi} \mu_A(x_i)$$

- Plays important role in fuzzy databases and information systems.

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II. Height : Normal vs. Subnormal

- Height of a fuzzy set is the highest membership value of its membership function

  $$\text{Height}(A) = \mathit{max}\ \mu_A(x_i)$$

- Normal Fuzzy Set – a fuzzy set with height 1
- Subnormal Fuzzy set – a fuzzy set whose height is less than 1.
- A subnormal fuzzy set contain only partial members but no full members.

III. Support and Alpha level cuts

- Support – set of elements whose degree of membership in A is greater than 0.

  $$Spt(A) = \{x \in U \mid \mu_A(x_i) > 0\}$$

- Alpha cut(α-level) – set of elements whose degree of membership in A is no less than α0.

  $$A\alpha_0 = \{x \in U \mid \mu_A(x) \geq \alpha_0\}$$

IV. Resolution Identity

- Reconstructing a membership function from its α-cuts is called resolution identity in fuzzy set theory.
- $A = \alpha_{0\,x} A\alpha_0 + \alpha_{1\,x} A\alpha_1 + \ldots\ldots + \alpha_{n\,x} A\alpha_n$

  Where $\alpha_{i\,x} A\alpha_i$ represents a fuzzy set

  $$\mu\, \alpha_{i\,x} A\alpha_i = \begin{cases} \mu_i & \text{if } \mu_A(x_i) \ge \alpha_i \\ 0 & otherwise \end{cases}$$

  and + represents the disjunction operator

V. Convex Fuzzy sets

- A fuzzy set is convex if its membership function does not have a "valley".
- Let U be the universe of discourse of a variable x. Let A be a fuzzy subset of U. The set A is convex if and only if

$$\mu_A(\lambda a + (1-\lambda) b) \geq \min\{\mu_A(a), \mu_A(b)\}$$

for all $a, b \in U$ and $0 \leq \lambda \leq 1$

i.e. the membership value of any given element in the interval {a.b} should be no less than the membership value of either end points