Linear Algebra

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Outline

- Linear Algebra Basics
- Matrix Calculus
- Singular Value Decomposition (SVD)
- Eigenvalue Decomposition
- Low-rank Matrix Inversion

Basic concepts

- Vector in R^n is an ordered set of n real numbers.
  - e.g. v = (1,6,3,4) is in R^4
  - A column vector: $\begin{pmatrix} 1 \\ 6 \\ 3 \\ 4 \end{pmatrix}$
  - A row vector: $\begin{pmatrix} 1 & 6 & 3 & 4 \end{pmatrix}$
- m-by-n matrix is an object in R^{mxn} with m rows and n columns, each entry filled with a (typically) real number:

$$\begin{pmatrix} 1 & 2 & 8 \\ 4 & 78 & 6 \\ 9 & 3 & 2 \end{pmatrix}$$

Basic concepts

Vector norms: A norm of a vector ||x|| is informally a measure of the "length" of the vector.

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

– Com -2 (Euclidean)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

– L$_{ir}$

$$\|x\|_\infty = \max_i |x_i|$$

Basic concepts

We will use lower case letters for vectors The elements are referred by x_i.

- Vector dot (inner) product:

$$x^T y \in \mathbb{R} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ x_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i .$$

If u•v=0, $||u||_2$!= 0, $||v||_2$!= 0 → *u and v are orthogonal*

If u•v=0, $||u||_2$ = 1, $||v||_2$ = 1 → *u and v are orthonormal*

- Vector outer product:

$$xy^T \in \mathbb{R}^{m\times n} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}$$

Basic concepts

We will use upper case letters for matrices. The elements are referred by $A_{i,j}$.

- Matrix product:

$$A \in \mathbb{R}^{m \times n} \qquad B \in \mathbb{R}^{n \times p}$$

$$C = AB \in \mathbb{R}^{m \times p}$$

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

e.g.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Special matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$ diagonal

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$ upper-triangular

$$\begin{pmatrix} a & b & 0 & 0 \\ c & d & e & 0 \\ 0 & f & g & h \\ 0 & 0 & i & j \end{pmatrix}$$ tri-diagonal

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}$$ lower-triangular

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ I (identity matrix)

Basic concepts

Transpose: You can think of it as

- "flipping" the rows and columns

OR

- "reflecting" vector/matrix on line

e.g. $\begin{pmatrix} a \\ b \end{pmatrix}^T = \begin{pmatrix} a & b \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A+B)^T = A^T + B^T$

Linear independence

- A set of vectors is linearly independent if none of them can be written as a linear combination of the others.
- Vectors $v_1,\ldots,v_k$ are linearly independent if $c_1v_1+\ldots+c_kv_k = 0$ implies $c_1=\ldots=c_k=0$

$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

e.g.

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(u,v)=(0,0), i.e. the columns are linearly independent.

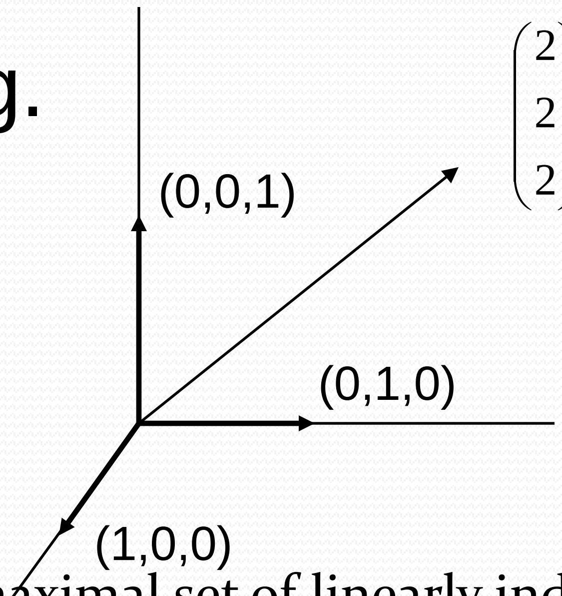
$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad x_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \quad x_3 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

x3 = −2x1 + x2

Span of a vector space

- If all vectors in a vector space may be expressed as linear combinations of a set of vectors $v_1,\dots,v_k$, then $v_1,\dots,v_k$ spans the space.
- The cardinality of this set is the dimension of the vector space.

e.g.



$$\begin{pmatrix}2\\2\\2\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

- A basis is a maximal set of linearly independent vectors and a minimal set of spanning vectors of a vector space

Rank of a Matrix

- rank(A) (the rank of a m-by-n matrix A) is

  The maximal number of linearly independent columns

  =The maximal number of linearly independent rows

  =The dimension of col(A)

  =The dimension of row(A)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$$

- If A is n by m, then
  - rank(A)<= min(m,n)
  - If n=rank(A), then A has full row rank
  - If m=rank(A), then A has full column rank

Inverse of a matrix

- Inverse of a square matrix A, denoted by A^{-1} is the *unique* matrix s.t.
  - $AA^{-1} = A^{-1}A = I$ (identity matrix)
- If A^{-1} and B^{-1} exist, then
  - $(AB)^{-1} = B^{-1}A^{-1}$,
  - $(A^T)^{-1} = (A^{-1})^T$
- For orthonormal matrices
- For diagonal matrices

$$\mathbf{A}^{-1} = \mathbf{A}^\top$$

$$\mathbf{D}^{-1} = \mathrm{diag}\{d_1^{-1}, \ldots, d_n^{-1}\}$$

Dimensions

	Scalar	Vector	Matrix
Scalar	$\frac{\mathrm{d}y}{\mathrm{d}x}$	$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \left[\frac{\partial y_i}{\partial x}\right]$	$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = \left[\frac{\partial y_{ij}}{\partial x}\right]$
Vector	$\frac{\mathrm{d}y}{\mathrm{d}\mathbf{x}} = \left[\frac{\partial y}{\partial x_j}\right]$	$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\mathbf{x}} = \left[\frac{\partial y_i}{\partial x_j}\right]$	
Matrix	$\frac{\mathrm{d}y}{\mathrm{d}\mathbf{X}} = \left[\frac{\partial y}{\partial x_{ji}}\right]$		

Ex

$$\frac{\partial \mathbf{x}^T \mathbf{a}}{\partial \mathbf{x}} = \frac{\partial \mathbf{a}^T \mathbf{x}}{\partial \mathbf{x}} = \mathbf{a}$$

$$\frac{\partial \mathbf{a}^T \mathbf{X} \mathbf{b}}{\partial \mathbf{X}} = \mathbf{a}\mathbf{b}^T$$

$$\frac{\partial \mathbf{a}^T \mathbf{X}^T \mathbf{b}}{\partial \mathbf{X}} = \mathbf{b}\mathbf{a}^T$$

$$\frac{\partial \mathbf{a}^T \mathbf{X} \mathbf{a}}{\partial \mathbf{X}} = \frac{\partial \mathbf{a}^T \mathbf{X}^T \mathbf{a}}{\partial \mathbf{X}} = \mathbf{a}\mathbf{a}^T$$

$$\frac{\partial \mathbf{x}^T \mathbf{B} \mathbf{x}}{\partial \mathbf{x}} = (\mathbf{B} + \mathbf{B}^T)\mathbf{x}$$

Singular Value Decomposition (SVD)

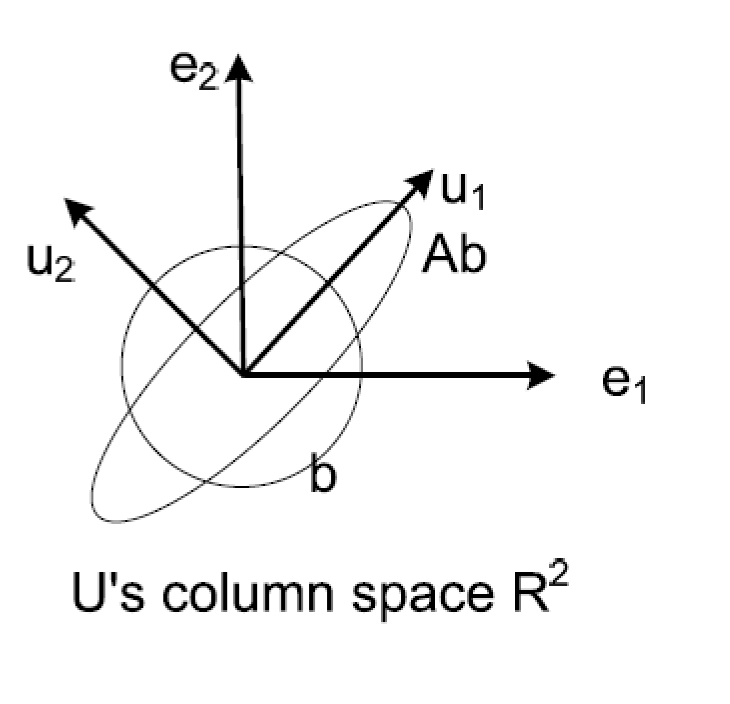
- Any matrix A can be decomposed as $A=UDV^T$, where
  - D is a diagonal matrix, with d=rank(A) non-zero elements
  - The fist d rows of U are orthogonal basis for col(A)
  - The fist d rows of V are orthogonal basis for row(A)

- Applications of the SVD
  - Matrix Pseudoinverse
  - Low-rank matrix approximation

Eigen Value Decomposition

- Any symmetric matrix A can be decomposed as $A=UDU^T$, where
  - D is diagonal, with d=rank(A) non-zero elements
  - The fist d rows of U are orthogonal basis for col(A)=row(A)

- Re-interpreting Ab
  - First stretch b along the direction of u_1 by d_1 times
  - Then further stretch it along the direction of u_2 by d_2 times



Low-rank Matrix Inversion

- In many applications (e.g. linear regression, Gaussian model) we need to calculate the inverse of covariance matrix X^TX (each row of n-by-m matrix X is a data sample)
- If the number of features is huge (e.g. each sample is an image, #sample n<<#feature m) inverting the m-by-m X^TX matrix becomes an problem
- Complexity of matrix inversion is generally $O(n^3)$
- Matlab can comfortably solve matrix inversion with m=thousands, but not much more than that

Low-rank Matrix Inversion

- With the help of SVD, we actually do NOT need to explicitly invert X^TX
  - Decompose $X=UDV^T$
  - Then $X^TX = VDU^TUDV^T = VD^2V^T$
  - Since $V(D^2)V^TV(D^2)^{-1}V^T=I$
  - We know that $(X^TX)^{-1}= V(D^2)^{-1}V^T$
  - Inverting a diagonal matrix D^2 is trivial

THANKS