Taylor Series

Dr Gaurav Verma
Asst Professor in Mathematics

Objectives

- Find a Taylor or Maclaurin series for a function.
- Find a binomial series.
- Use a basic list of Taylor series to find other Taylor series.

Taylor Series and Maclaurin Series

Taylor Series and Maclaurin Series

The theorem gives the form that *every* convergent power series must take.

THEOREM 9.22 The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \sum a_n(x - c)^n$ for all x in an open interval I containing c, then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots.$$

The coefficients of the power series in Theorem 9.22 are precisely the coefficients of the Taylor polynomials for $f(x)$ at c. For this reason, the series is called the **Taylor series** for $f(x)$ at c.

Taylor Series and Maclaurin Series

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the **Taylor series for $f(x)$ at c**. Moreover, if $c = 0$, then the series is the **Maclaurin series for f**.

Example 1 – *Forming a Power Series*

Use the function $f(x) = \sin x$ to form the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

and determine the interval of convergence.

Solution:

Successive differentiation of $f(x)$ yields

$f(x) = \sin x$ $\quad$ $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $\quad$ $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ $\quad$ $f''(0) = -\sin 0 = 0$

$f^{(3)}(x) = -\cos x$ $\quad$ $f^{(3)}(0) = -\cos 0 = -1$

Example 1 – *Solution*

cont'd

$f^{(4)}(x) = \sin x$ $f^{(4)}(0) = \sin 0 = 0$

$f^{(5)}(x) = \cos x$ $f^{(5)}(0) = \cos 0 = 1$

and so on.

The pattern repeats after the third derivative.

Example 1 – *Solution*

cont'd

So, the power series is as follows.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = 0 + (1)x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{(-1)}{7!}x^7 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

By the Ratio Test, you can conclude that this series
converges for all *x*.

Taylor Series and Maclaurin Series

You cannot conclude that the power series converges to sin *x* for all *x*.

You can simply conclude that the power series converges to some function, but you are not sure what function it is.

This is a subtle, but important, point in dealing with Taylor or Maclaurin series.

To persuade yourself that the series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

remember that the derivatives are being evaluated at a single point.

Taylor Series and Maclaurin Series

It can easily happen that another function will agree with the values of $f^{(n)}(x)$ when $x = c$ and disagree at other x-values.

If you formed the power seri
the function shown in Figure
you would obtain the same s
as in Example 1.

You know that the series con
for all x, and yet it obviously
converge to both $f(x)$ and $\sin x$ for all x .

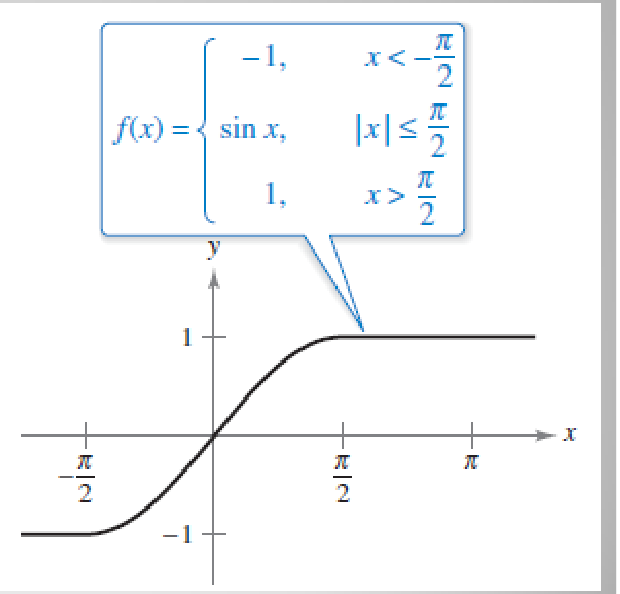


Figure A

Taylor Series and Maclaurin Series

Let *f* have derivatives of all orders in an open interval *I* centered at *c*.

The Taylor series for *f* may fail to converge for some *x* in *I*. Or, even if it is convergent, it may fail to have *f*(*x*) as its sum.

Nevertheless, Theorem tells us that for

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x),$$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

where

Taylor Series and Maclaurin Series

Note that in this remainder formula, the particular value of *z* that makes the remainder formula true depends on the values of *x* and *n*. If $R_n \to 0,$ then the next theorem tells us that the Taylor series for *f* actually converges to *f* (*x*) for all *x* in *I.*

THEOREM 9.23 Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n.$$

Example 2 – *A Convergent Maclaurin Series*

Show that the Maclaurin series for $f(x) = \sin x$ converges to $\sin x$ for all x.

Solution:

You need to show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdot\ \cdot\ \cdot + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdot\ \cdot\ \cdot$$

is true for all x.

Example 2 – *Solution*

cont'd

Because

$$f^{(n+1)}(x) = \pm\sin x$$

or

$$f^{(n+1)}(x) = \pm\cos x$$

you know t $\left|f^{(n+1)}(z)\right| \le 1$ for every real number *z*.

Therefore, for any fixed *x*, you can apply Taylor's Theorem to

$$0 \le \left|R_n(x)\right| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!}.$$

Example 2 – *Solution*

cont'd

From the discussion regarding the relative rates of convergence of exponential and factorial sequences, it follows that for a fixed x

$$\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

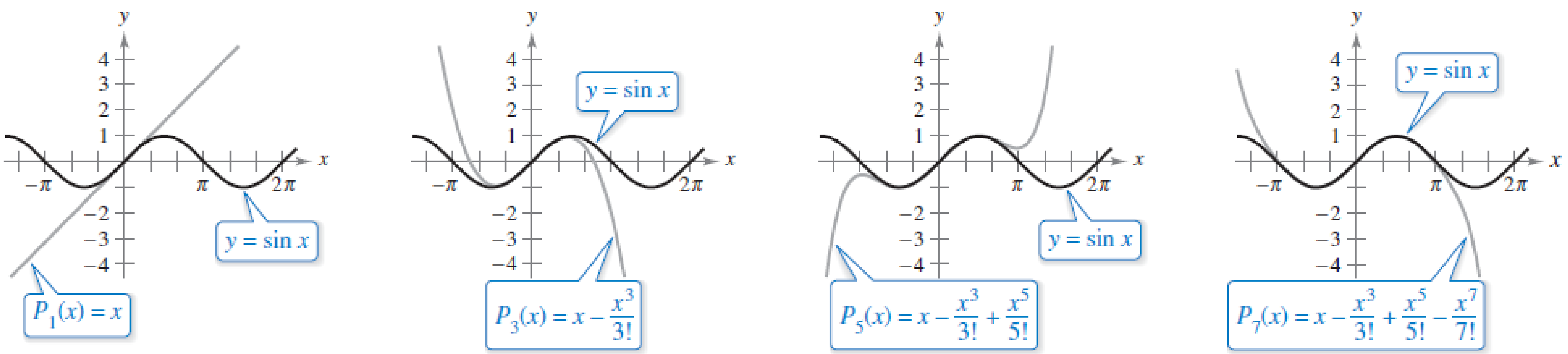
Finally, by the Squeeze Theorem, it follows that for all x,

$R_n(x) \to 0$ as $n \to \infty$.

So, by Theorem 9.23, the Maclaurin series for sin x converges to sin x for all x.

Taylor Series and Maclaurin Series

Figure B visually illustrates the convergence of the Maclaurin series for sin *x* by comparing the graphs of the Maclaurin polynomials $P_1(x)$, $P_3(x)$, $P_5(x)$, and $P_7(x)$ with the graph of the sine function. Notice



As n increases, the graph of P_n more closely resembles the sine function.

Figure B

Taylor Series and Maclaurin Series

GUIDELINES FOR FINDING A TAYLOR SERIES

1. Differentiate $f(x)$ several times and evaluate each derivative at c.

$$f(c), f'(c), f''(c), f'''(c), \cdot \cdot \cdot, f^{(n)}(c), \cdot \cdot \cdot$$

Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdot \cdot \cdot + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdot \cdot \cdot .$$

3. Within this interval of convergence, determine whether the series converges to $f(x)$.

Deriving Taylor Series from a Basic List

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \le x \le 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1$*

* The convergence at $x = \pm 1$ depends on the value of k.

Example 6 – *Deriving a Power Series from a Basic List*

Find the power serie $f(x) = \cos\sqrt{x}.$

Solution:

Using the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

you can replace $\sqrt{x}$ by to obtain the series

$$\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges for all *x* in the c $\cos\sqrt{x}$ in of —that is, for $x \geq 0$.