LebesgueTheory I

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Preposition: A subject O of $\mathbb{R}$ is open if and only if it is union of at the most of countable number of disjoint open intervals.

Proof: If O is a countable union of open intervals then clearly O is open.

Conversely, suppose that O is an open set.

Let $x \in O$ then there exists y such that $(x,y) \subseteq O$

Let $b = \sup\{ y \mid (x,y) \subseteq O\}$

Similarly, there exists z such that $(z,x) \subseteq O$

Let $a = \inf\{z \mid (z,x) \subseteq O\}$

Define

$I_x = (a,b)$ then $x \in I_x$

And $I_x \subseteq O$. Since if $w \in I_x$, $w \neq x$

Then either $w<x$ or $w>x$ and $a < w < b$.

Next, if $a \in O$, then $(a-\varepsilon, a+\varepsilon) \in O$ for some $\varepsilon>0$

and so $(a-\varepsilon, x) \subseteq O$.

This contradicts the definition of a.

Therefore $a \notin O$. Similarly, $b \notin O$.

Now, $O = \bigcup_{x \in O} Ix$

To prove that the result we need to show that the collection $\{I_x \mid x \in O\}$ has at the most countable number of open intervals which are mutually disjoint.

To show that this collection consists of disjoint open intervals, we show that for $x,y \in O$, $x \neq y$ either $I_x = I_y$ or $I_x \cap I_y = \emptyset$

Let $I_x = (a,b)$ and $I_y = (c,d)$

Let $w \in I_x \cap I_y$

Then $a<w<d$ implies that $a<d$.

Since $a<d$ and $I_y = (c,d)$ and $a \notin O$, $a \leq c$.

Similarly, $c<w<b \Rightarrow c<b$ and so $c \leq a$

Therefore $c=a$.

Similar arguments imply that $b=d$.

Hence $I_x \cap I_y \neq \emptyset \Rightarrow I_x = I_y$

Thus the collection, $C = \{I_x \mid x \in O\}$ consists of disjoint open intervals.

Let q_x be a fixed rational number in I_x

Then $I_x \rightarrow q_x$ is one-one mapping from C to Q.

Therefore, C consists of at most countable numbers of distinct open intervals.

This completes the proof.

Lindel of Covering

Preposition: Let e= {O_α | $\alpha \in \Lambda$} be a collection of open intervals in ℝ. Then e has a countable sub- collection {O_i | i= 1, 2, _ _ _} such that

$$\bigcup_{\alpha \in \Lambda} O\alpha = \bigcup_{i=1}^{\infty} Oi$$

Proof: Let x ∈ $\bigcup_{\alpha \in \Lambda} O\alpha$. Then x ∈ O_α for some $\alpha \in \Lambda$ and so there exists an open interval I_x such that $x \in I_x \subseteq O_\alpha$

Let J_x be an open interval with rational end points

such that $x \in J_x \subseteq I_x$.

Therefore, $\bigcup_{\alpha} O_\alpha = \bigcup_{x \in \cup O_\alpha} J_x$

Since, $\{J_x \mid x \in O_\alpha\}$ is a countable collection.

Write $\{J_x \mid x \in O_\alpha\} = \{J_1, J_2, \_\_\_\}$

Let $O_k \in e$ such that $J_k \in O_k$

Then $\bigcup_{\alpha \in \Lambda} O_\alpha = \bigcup_{k=1}^{\infty} O_K$

Unit III: Lebesgue Theory

Let I_1 be an interval in $\mathbb{R}$.

Define the length of I.

$$l(I) = \begin{cases} \infty & \text{if } I \text{ is unbounded} \\ \text{Sup } I - \text{Inf } I & \text{if } I \text{ is bounded} \end{cases}$$

Bounded intervals in $\mathbb{R}$ are [a,b], (a,b], (a,b), [a,b)

for $a,b \in \mathbb{R}$.

The length of these intervals is b-a.

We define (Lebesgue) outer Measure.

Let $A \subseteq \mathbb{R}$ and let $\{I_i \mid i = 1, 2, \_\ \_\}$ be the set of open intervals in $\mathbb{R}$ such that

$$A = \bigcup_{i=1}^{\infty} I_i$$

Define $m^* A = \inf \{ \sum_{i=1}^{\infty} l(I_i) \mid A \subseteq \bigcup_{i=1}^{\infty} I_i \}$

Where the inf is taken over all at the most countable number of open intervals which cover I.

If $A \subseteq \mathbb{R}$;

$0 \leq m^*A < \infty$

1) Outer Measure of A is non- negative.

2) If $A \subseteq B \subseteq \mathbb{R}$

$m^*A \leq m^*B$

($\because$ Every open covering of B is open covering of A)

As set increases, its measure also increases.

$\Rightarrow$ m* is non- negative and monotone function.

$m^* : P(\mathbb{R}) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$

3) For any $x \in \mathbb{R}$,

Define $A+x = \{a+x \mid a \in A\}$

Then, $m^*(A+x) = m^*A$,

$l(I_n + x) = l(I_n)$

If $\{I_n\}_{n=1}^{\infty}$ is an open covering of A.

Then, $\{I_n+x\}_{n=1}^{\infty}$ is an open covering of A+x.

$$A+x \subseteq \bigcup_{n=1}^{\infty} I_n+x$$

$$m^*(A+x) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n+x\right)$$

$$= \sum_{n} l(I_n+x)$$

$$= \sum_{n} l(I_n)$$

4) $m^*A < \infty$ for any bounded set A.

Proof: Let a= inf A and b= sup A

Then for any $\varepsilon>0$,

$A \subseteq (a - {}^{\varepsilon}/_{2} , b + {}^{\varepsilon}/_{2})$

And so by definition,

$m^*A \leq l (a - {}^{\varepsilon}/_{2} , b + {}^{\varepsilon}/_{2})$

$= b – a + \varepsilon$

Then, $m^*A < \infty$

5) $m^*A = 0$ if A is countable.

Let $A = \{ a_1 , a_2 , \_ \_ \}$

For any $\varepsilon > 0$, consider the intervals

$I_n = (a_n - \varepsilon/2^{n+1} , a_n + \varepsilon/2^{n+1})$

Then for each n, $a_n \in I_n$

So, $A \subseteq \bigcup_{n=1}^{\infty} I_n$

Therefore,

$$m^*A \leq \sum_{i=1}^{\infty} l\,(I_n)$$

$$= \sum_{i=1}^{\infty} \varepsilon/2^n$$

$$= \varepsilon$$

Hence for any $\varepsilon > 0$,

$m^*A \leq \varepsilon$

Since ε is arbitrary number,

We have, $m^*A = 0$

Definition: A set with outer measure zero is a null set.

Example: Q is countable.

∴ Its outer measure = 0

$m^*Q = 0$ ∴ **It is a null set**

(Q is unbounded but it is having finite measure so if a set is unbounded then measure may be finite or not)

Metric set is said to separate if it has a countable dense subset.

An uncountable set with measure zero is Q and this set is called cantor set.

For cantor set,

$m^*A \leq (2/3)^n \;\forall\; n$

As $n \to \infty$

$m^*A = 0$

∴ Cantor set has measure zero.

It is perfect so it is uncountable.

∴ A cantor set is compact perfect.

So, it is uncountable

And its measure is zero.

Example: Let I be an interval then m*I = *l* (I)

Proof: Assume first that the result holds for compact intervals

i.e. Intervals of type [a,b], $a,b \in \mathbb{R}$

If I is bounded and

a= inf I , b= sup I

$a,b \in \mathbb{R}$, we have for any $\varepsilon > 0$

$[a + {}^{\varepsilon}/_{2}\ , b - {}^{\varepsilon}/_{2}\ ] \subseteq I \subseteq [a,b]$

And so

$m^* [a + {}^{\varepsilon}/_{2}\ , b - {}^{\varepsilon}/_{2}\ ] \leq m^*I \leq m^* [a,b]$

$b - a - \varepsilon \leq m^*I \leq b - a$

Since ε is arbitrary

$\Rightarrow m^*I = b-a$

$m^*I = l(I)$

If I is unbounded then for any positive integer n,

I has an interval J such that $l(J) = n$

Thus $J \subseteq I$

$\Rightarrow m^*I > m^*J = l(J) = n$

Therefore

$m^*I = \infty = l(I)$

Let I be a compact interval and let $\varepsilon > 0$

Then $\exists$ an open covering $\{ I_n \}_{n=1}^{\infty}$ such that

$$m^*(I) + \varepsilon > \sum_{n=1}^{\infty} l(I_n)$$

Since I is compact, this open covering has a finite subcovering.

Let $J_i = (a_1, b_1), \_\ \_\ \_\ J_m = (a_m, b_m)$ be a subcovering of I such that

$I \cap J_i \neq \phi \ \forall\ i$

And $a_1 < a_2 < \_ \_ < a_m$

Since an interval is connected,

The intervals J_i 's are overlapping.

Therefore,

$$\sum_{n=1}^{m} l(J_i) \geq l(I) = b - a$$

Thus,

$$m^*(I) + \varepsilon \geq \sum_{n=1}^{\infty} l(I_n)$$

$$\geq \sum_{n=1}^{m} l\ (J_i) \geq l(I)$$

Therefore, $m^*(I) \geq l(I)$

Also as $I = [a,b]$

$\Rightarrow I \subseteq [a - {}^{\varepsilon}/_{2}\ ,\ b + {}^{\varepsilon}/_{2}\ ]$

$\Rightarrow m^*I \leq b - a = l(I)$

$\Rightarrow m^*I = l(I)$

Outer Measure

Theorem: Let (A_n) be a sequence of subsets of $\mathbb{R}$ then $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^* A_n$

Proof:

If $m^* A_n = \infty$ then it is obvious; if $m^* A_n < \infty$

Let $\varepsilon>0$, for each n, $\exists$ an open interval such that $A_n \subseteq \bigcup_{k=1}^{\infty} I_{n,k}$

And $m^* A_n + \varepsilon/2^n > \sum_{k=1}^{\infty} l(I_{n,k})$

Now, $\{I_{n,k}\}_{n,k}$ is a sequence of open intervals such that

$$\left(\bigcup_{n=1}^{\infty} \mathbf{A_n}\right) \subseteq \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} I_{n,\kappa}$$

Then,

$$m^* \left(\bigcup_{n=1}^{\infty} \mathbf{A_n}\right) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} l(I_{n,k})$$

$$< \sum_{n=1}^{\infty} m^* \mathbf{A_n} + \sum_{n=1}^{\infty} \varepsilon/2^n$$

$$\leq \sum_{n=1}^{\infty} m^{*}A_n + \varepsilon$$

Hence, $m^{*} \left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^{*}A_n$

Example:

On open interval (0,1) for $x, y \in (0,1)$ define

$x \sim y$ if $x - y \in Q$.

Then ~ is an equivalence relation.

Let A be the set consisting of exactly one element from each equivalence class.

Then $A \subseteq (0,1)$

And so $m^*A < \infty$ $(\because A \text{ is bdd})$

A_1 , A_2 are equivalence classes.

Therefore,

$\cup A_i = (0,1)$

(0,1) is uncountable

Since $m^*(0,1) = 1 \neq 0$

Therefore, by theorem

(0, 1) is uncountable.

$(0,1) = A_1 \cup A_2$ _ _

Each of A_1 , A_2 are countables

Collection of sets → points both uncountable

Let $D = (-1,1) \cap Q$

Let $B = D + A$

Since D is countable with $D = \{r_1, r_2, \_ \_\}$

So that

$$B = \bigcup_{n=1}^{\infty} (r_n + A)$$

$(0,1) \subseteq B \subseteq (-1, 2)$

Thus, $1 \leq m^*B \leq 3$

Let $A_n = r_n + A$

For $n \neq m$,

$A_n \cap A_m = \phi$

Let $x \in A_n \cap A_m$

$x = r_n + a = r_m + a'$

r_n and r_m are rationals.

So a and a' differ by rational.

But we are taking one element from each equivalence class and a and a' are equivalent.

$\Rightarrow a = a'$

$\Rightarrow a, a' \in A$

$\Rightarrow r_n = r_m$, $n=m$

Also $m^* A_n = m^*A$

Now (A_n) is sequence of disjoint and distinct sets

with finite measure

Such that $B= \bigcup_{n=1}^{\infty} A_n$

For these sets,

$$m^*B \neq \sum_{n=1}^{\infty} m^* A_n$$

First note that $m^*A \neq 0$

As then $m^*B = 0$

$$\sum_{n=1}^{\infty} m^* A_n = \infty$$

But, $B = \bigcup_{n=1}^{\infty} A_n$

m^*B is finite

Therefore,

$$m^*B \neq \sum_{n=1}^{\infty} m^* A_n$$

A subset E of $\mathbb{R}$ is called Lebesgue Measurable

if $m^*A = m^*(A \cap E) + m^*(A \cap E^c)$

For every $A \subseteq \mathbb{R}$,

(I) If E is measurable then so is $\mathbf{E^c}$

Since for $A, B \subseteq \mathbb{R}$

$m^*(A \cup B) \leq m^*A + m^*B$

$A = (A \cap E) \cup (A \cap E^c)$

So, $m^*A \leq m^*(A \cap E) + m^*(A \cap E^c)$

For showing that E is measurable, we need to check that

$m^*A \geq m^*(A \cap E) + m^*(A \cap E^c)$

(II) If $m^* E = 0$ then E is measurable.

Let $A \subseteq \mathbb{R}$

Since $E \cap A \subseteq E$

$m^* (A \cap E) = 0$

Now, $(A \cap E^c) \subseteq A$

$\Rightarrow m^* (A \cap E^c) \leq m^* A$

Thus, $m^*A \geq m^*(A \cap E) + m^*(A \cap E^c)$

Hence E is measurable.

(III) If E is measurable then so is E+x for any $x \in \mathbb{R}$

$$\Rightarrow m^*A = m^*(A-x) = m^*((A-x) \cap E) + m^*((A-x) \cap E^c)$$

$$= m^* ((A\text{-}x) \cap E + x) + m^* ((A\text{-}x) \cap E^c + x)$$

$$= m^* (A \cap E + x) + m^* (A \cap (E + x)^c)$$

Hence,

E + x is measurable.

(IV) If E and F are measurable then $E \cup F$ is also measurable.

$\Rightarrow$ If E, F are measurable then,

$$E \cap F = (E^c \cup F^c)^c$$

$$E \setminus F = E \cap F^c$$

E, F are measurable $\Rightarrow E \cup F$ is also measurable.

To prove this, we need to verify that

$$m^*A = m^*(A \cap (E \cup F) + m^*(A \cap (E \cup F)^c$$

For any $A \subseteq \mathbb{R}$,

$$m^*A = m^*(A \cap (E \cup F) + m^*(A \cap (E^c \cap F^c)$$

Now,

$$(A \cap (E \cup F) \cap E = A \cap E$$

$$(A \cap (E \cup F)) \cap E^c = A \cap F \cap E^c$$

$$m^*A = m^*(A \cap E) + m^*(A \cap E^c)$$

$$= m^*(A \cap E) + m^*(A \cap E^c \cap F) +$$

$$m^*(A \cap E^c \cap F^c)$$

$$= m^*(A \cap (E \cup F) \cap E) + m^*(A \cap F \cap E^c)$$

$$+ m^*(A \cap (E \cup F)^c$$

$$= m^* (A \cap (E \cup F) \cap E) +$$

$$m^* (A \cap (E \cup F) \cap E^c) + m^* (A \cap (E \cup F)^c)$$

$$= m^* (A \cap (E \cup F)) + m^* (A \cap (E \cup F)^c)$$

Now,

$$m^* (A \cap (E \cup F)) = m^* (A \cap E) +$$

$$m^* (A \cap F \cap E^c)$$

Provided E is measurable.

If $E \cap F = \phi$, E, F measurable

Then $F \subseteq E^c$

$m^* (A \cap (E \cup F)) = m^* (A \cap E) + m^* (A \cap F)$

In particular,

If $A = E \cup F$

$m^* (E \cup F) = m^*E + m^*F$

Thus, we can say

If E and F are measurable and $E \cap F = \phi$

Then,

$m^* (A \cap (E \cup F)) = m^* (A \cap E) + m^* (A \cap F)$

$\forall A \subseteq \mathbb{R}$

And, $m^* (E \cup F) = m^*(E) + m^* (F)$

Now, inductively, if E_1 , E_2 ,_ _ _, E_n are mutually disjoint measurable sets then,

$$m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{n=1}^{\infty} m^* E_i$$