Probability Theory-I

Introduction

Definitions of Probability

We shall consider following two definitions of Probability

- Mathematical or a priori or classical.
- Statistical or empirical

Mathematical or priori or classical

- If there are 'n' exhaustive, mutually exclusive and equally likely cases and m of them are favourable to an event A, the probability of A happening is defined as the ratio m/n.
- Formula is expressed as:

$$P(A) = \frac{m}{n}$$

$$= \frac{\text{number of favourable outcomes}}{\text{total no of possible outcomes}}$$

Cont…

This definition is due to ‘Laplace’. Thus probability is concept which measures numerically the degree of certainty or uncertainty of the occurrence of an event.

Example

For example, the probability of randomly drawing taking from a well-shuffled deck of 52 cards is 4/52. Since 4 is the number of favourable outcomes

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(i.e. 4 kings of diamond, spade, club and heart) and 52 is the number of total outcomes (the total number of cards in the deck).

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If A is any event of sample space having probability P, the clearly, P is a positive number (expressed as a fraction or usually as a decimal) not greater than unity. That is probability P is less than or equal to 1 and greater than equal to zero.

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Since the number of cases not favourable to A are (n-m), the probability q that event A will not happen is (n-m)/n.

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Note that the probability q is nothing but the probability of the complementary event A i.e.

Thus P(Â)=1-P(A)

So P(A)+P(Â)=1.

Relative Frequency Definition

- The classical definition of probability has a disadvantage i.e. the words ‘equally likely’ are vague. In fact, since these words seem to be synonymous with “equally probable”. This definition is circular as it I defining (in terms) of itself.

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Therefore, the estimated or empirical probability of an event is taken as relative frequency of the occurrence of the event when the number of observations is very large.

Statistical Definition

- If trials are to be repeated a greater number of times under essentially the same condition then the limit of the ratio of the number of times that an event happens to the total number of trials, as the number of trials increases indefinitely is called the probability of the happening of the event.

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- It is assumed that the limit exists and finite uniquely. Symbolically

$P(A)=p=\lim_{n \to \infty} \frac{m}{n}$

Provided it is finite and unique.

Observation

- The two definitions are apparently different but both of them can be reconciled the same sense

Examples

1. Two dice are rolled. Find the probability that the score on the second die is greater than the score on the first die.

Solution of Example 1.

- When two dice are rolled the sample space is as follow:

S={(1,1), (1,2),(1,3),(1,4),(1,5),(1,6),(2,1)………(6,6)}

n(S)=36.

Cont…

Event A: The score on the second die is greater than the score on first die.

Cont…

i.e.

A={(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)}

n(A)=15

Therefore, p(A)=n(A)/n(S)=15/36=5/12=0.42

Example 2.

What is the chance that a leap year selected at random will contain 53 Sundays?

Solution of Example 2.

A leap year has 52 weeks and 2 days more.

The two days can be:

Monday-Tuesday

Tuesday-Wednesday

Wednesday-Thursday

Thursday-Friday

Friday-Saturday

Cont..

Saturday-Sunday

Sunday-Monday.

There are 7 outcomes and 2 are favourable to our event.

Now 53 Sundays in a leap year,

$P(A)=n(A)/n(S)=2/7=0.29$.

Expressing Probability Mathematically

We can express the probability of an event mathematically in a number of ways, but the most common is a decimal between 0 and 1. Thus, a probability of 1 is certainty that event will occur.

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A probability of 0 is certainty that event will not occur. Thus all the probabilities for an event are expressed as a decimal between 0 and 1.

Cont.. ..

The probability that a single fair coin toss will produce a head on top is expressed as follow:

$p=.5$. This we might also express as meaning that the probability is 5 out of 10 or 50%.

Cont…

The decimal notation is the most common and standard.

Probability Theory and Observation

One of the major applications of probability is to check the theoretical possibilities of a particular outcome with the observed result of an event.

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If we find that the observed frequency of a particular occurrence departs widely from theoretical probabilities, sometimes we have good reason to investigate the event more carefully.

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For example, in a fair toss of a coin, we theoretically calculate p as .5 for head to appear on any single toss.

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But what is the probability that someone will throw the coin four times in a row and have the outcome head each time?

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We can calculate the probability by multiplying the probabilities together. Since there are four tosses and the probability of a head is .5 on each of these the probability of a sequence of four heads is as follow:

$$p=.5*.5*.5*.5=.0625$$

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In other words, in a sequence of four consecutive coin tosses, probability theory indicated that in 625 cases out of 10,000 four heads will come up in a row.

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Thus, if you were making a bet in a number of different trials of four consecutive coin tosses that four heads would appear in a row. You could theoretically expect to win 1 out of every 16 trials and lose 15 out of 16 trials.

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Now this figure of 1 in 16 ($p=.0625$) does not claim that in every 16 trials of a four-coin-toss sequence, one result will always be four heads.

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The theory states that, on average, in a sequence of four-toss trials, the most likely outcome is that there will be four heads appearing in 1 out of 16 trials.

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But in any particular series of four-toss trials, it is quite possible for more than sixteen four-toss trials to happen before the first four-head result appear or for a four-head result to occur before sixteen trials are completed or more than once in one sixteen four-toss sequence.

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As we shall see, the more times one repeats the four toss sequence, the closer the observed result will come to the theoretically calculated result.

Cont…

Thus, given that one four-head sequence every sixteen throws is the most likely outcome of many four-toss sequence, for the game to be truly fair, you should put up $1 against your opponent’s $15 for each four-coin sequence.

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With a great many four-toss trials , you and your opponent should come out more or less equal.

Cont….

Hence, if you are betting $1 that four heads will appear in a four toss sequence of coins and if you opponent is unwilling to put up $15, you not undertake the bet. If she is willing to put up more than $15, they should take the bet.

The Fundamental Basis of Probability Theory

The basis upon which the mathematical treatment of probability rests is as follows: we assume that the phenomenon we wish to investigate can be described in terms of a given number of different possible but equally probable outcomes.

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For instance, in a simple coin toss there are two equally probable outcomes, heads or tails. In one roll of a die, there are six equally probable outcomes, represented by the six sides of the die, each with a number of dots from 1 to 6.

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In drawing a card from a well shuffled normal deck with no jokers, we know that there are four equally probable outcomes for the suit of the drawn card (hearts, diamonds, spades, or clubs), there are two possible outcomes for the colour

Cont…

of the card (red or black), and there are thirteen possible outcomes for the value of the card . Naturally we are assuming that in all these trials the procedure is fair.

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Multiple outcomes heads in consecutive coin tosses or so many sixes in consecutive rolls of a single dice can be calculated from the probability of a single event. Events which cannot be described in

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terms of equally probable outcomes are beyond the scope of probability theory discussed here.

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The probability of any specific outcome is given by the number which is equal to the number of possibilities which involve that outcome divided by the total number of possibilities . For example, in a simple coin

Cont..

toss , there is only one possibility that a head will appear, and the total number of possible outcomes is two. Thus, the probability that a head will appear is 1 divided by 2 or 5.

Cont..

In a roll of a single die, there is one chance that a six will appear, and there are six possible outcomes. Therefore the probability of a six appearing on a single roll of a normal die is

Cont…

1 divided by 6. If we draw a card from a normal full deck, there is only one chance that a heart will be on the card, and there are four possibilities foe different suits. Therefore the probability of drawing a heart from a full deck is 1/4 .

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What is the probability that on a single roll of a die, the result will be an even number? Well, there are three possible even outcomes and there are six possible outcomes. Thus, the probability that on a single roll of a die the result will be an even number of dots is 3/6 .

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More complicated outcomes are more complicated, but we can work out the possible outcomes for such events and calculate the probabilities involved these.

Randomness

Random phenomenon outcome cannot be predicted, but that has a regular distribution over many repetitions.

Probability proportions of times an event occur in many repeated events of a random phenomenon. Independent events ..outcome of an event does not influence the outcome of any other event.

Example of Independent event

Rolling a single die twice. The result on the second roll is independent of the first

Example of event not independent

Picking two cards from a well shuffled deck, one at a time, without replacement. (If you pick a card and do not put it back in the and reshuffled, then the probability of a red card on the second pick is dependent upon what the first card happened to be.)