Basic Definitions

Partition

A partition of an interval is a division of an interval into several disjoint sub-intervals, or we say Let $[a, b]$ be an interval of real numbers. A partition $\mathcal{P}$ is defined as the ordered n-tuples of real numbers $\mathcal{P} = (x_0, x_1, ...x_n)$ such that

$$a = x_0 < x_1 < ...x_n = b.$$

Norm

The norm of a partition $\mathcal{P}$ is defined as

$$\|\mathcal{P}\| = \sup\{x_i - x_{i-1}\}_{i=1}^{n}.$$

Jordan Arc

A continuous arc without multiple points is called *Jordan Arc* without multiple points is called jordan arc i.e, an arc that satisfies only one value of t.

Continuous Jordan Curve

A continuous Jordan Curve consists of chain of finite number of continuous arcs.

Contour

A contour is a continuous chain of finite number of regular arcs if A be the starting point of the first arc and B be the end point of last arc. Then the integral along such a curve can be written as,

$$\int_{AB} f(z)dz.$$

Regular Arc

If in addition to the definition of jordan arc, $\psi'(t)$ and $\phi'(t)$ arc also continuous within range

$$\alpha \leq t \leq \beta,$$

then that jordan arc is called regular arc.
The contour is said to be closed if the starting point A coincide with end point of the last arc. The integral along such a closed contour is written as,

$$\int_C f(z)dz$$

and is real as $f(z)$ taken on closed contour C.

Rectifiable Curve

Let the equation of arc '$\mathcal{L}'$ of plane curve be,

$$x = \phi(t), \quad y = \psi(t); \quad \alpha \leq t \leq \beta.$$

Divide the interval (α, β) into finite number of sub-intervals,

$$[t_0, t_1], [t_1, t_2], ...[t_{n-1}, t_n],$$

where $\alpha = t_0 < t_1 < t_2 < .... < t_{n-1} < t_n$. Let $z_0, z_1, ..., z_n$ be the points on the curve corresponding to $t_0, t_1, t_2, ...t_n$. We join each of the $z_0, z_1, ...z_{n-1}$ to next point by a straight line. Thus, we obtain a polygon .

Riemann Definition of Integration

Let $f(z)$ be a function complex variable be continuous in a domain $\mathcal{D}$ and a, b be two points in this domain, then the integral of $f(z)$ from a to b be defined as follow: Let $\mathcal{C}$ be any rectifiable curve lying entirely on $\mathcal{D}$ so that $f(z)$ be continuous on domain $\mathcal{C}$. Let $\mathcal{P} = \{a = z_0, z_1, z_2, ...z_n = b\}$ be any partition of $\mathcal{C}$ into n segments selected arbitrarily along the curve. On each segment joining z_{k-1} to z_k choose a point ξ_k. Consider the sum

$$S = \sum_{j=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{j=1}^{n} f(\xi_k)\Delta z_k.$$

Cont...

Let Δ be the length of the longest chord Δz_k. Let the number of subdivisions n approach infinity in such a way that the length of the longest chord approaches to zero. The sum S will then approaches to a limit which does not depend upon the length of subdivisions and is called the line integral of $f(z)$ from a to b along the curve $\mathcal{C}$:

$$\int_a^b f(z)dz = \lim_{\Delta \to 0} \sum_{k=1}^{infty} f(\xi_k)\Delta z_k.$$

Theorem

If F is a function such that $dF(z)/dz = f(z)$ at each point of $\mathcal{C}$, then

$$\int_a^b = F(b) - F(a).$$

Connection between Real and Complex Line Integral

Consider the line integral

$$\int_a^b P(x,y)dx + Q(x,y)dy.$$

If $P(x,y)dx + Q(x,y)dy$, or we say $Pdx + Qdy$, is an exact differential equation i.e. if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

there is a function $\phi(x,y)$ such that $d\phi = Pdx + Qdy$ and the line integral is equal to the change of $\phi(x,y)$ along the curve from point A to B . Moreover, if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere in a simply connected region, then the value of the line integral between two points of the region is path independent.

Real and complex line integral are connected as follow:

If $f(z) = u(x,y) + iv(x,y)$ be a complex valued function, then

$$I = \int_a^b f(z)dz = \int_a^b (u(x,y) + iv(x,y))(dx + idy)$$

$$= \int_{a=(x_0,y_0)}^{b=(x_n,y_n)} u(x,y)dx - v(x,y)dy + i \int_{a=(x_0,y_0)} u(x,y)dy + v(x,y$$

$$= \int_a^b udx - vdy + i \int_a^b vdx + udy.$$

Properties of Complex Line Integral

If $f(z)$ and $g(z)$ are integrable along curve $\mathcal{C}$, then

1. $\int_{\mathcal{C}} (f(z) + g(z))\, dz = \int_{\mathcal{C}} f(z)dz + \int_{\mathcal{C}} g(z)dz.$
2. $\int_{\mathcal{C}} cf(z)dz = c\int_{\mathcal{C}} f(z)dz, \quad \text{where c is any constant.}$
3. $\int_a^b f(z)dz = -\int_b^a f(z)dz.$
4. $\int_a^b f(z)dz =$

$\int_a^q f(z)dz + \int_q^b f(z)dz, \quad \text{where points a, q, b are in} \mathcal{C}.$

Upper Bound of a Complex Line Integral

Theorem

If a function $f(z)$ is continuous on a contour $\mathcal{C}$ of length $\mathcal{L}$ and and if M is upper bound of $|f(z)|$ on $\mathcal{C}$, then

$$\left| \int_{\mathcal{C}} f(z) dz \right| \leq M\mathcal{L}.$$

PROOF

Divide the contour $\mathcal{C}$ into n parts by means of the points z_0, z_1, z_2,.....z_n. We choose a point ξ_k on each arc joining z_{k-1} to z_k, then

$$\int_{\mathcal{C}} f(z)dz = \lim_{n\to\infty} \sum_{k=1}^{n} (z_k - z_{k-1}) f(\xi_k). \tag{1}$$

We know that modulus of sum of n complex numbers is less than equal to the sum of the modulus of these n complex numbers, therefore

$$\left| \sum_{k=1}^{n} (z_k - z_{k-1}) f(\xi_k) \right| \leq \sum_{k=1}^{n} |(z_k - z_{k-1}) f(\xi_k)|$$

$$= \sum_{k=1}^{n} |z_k - z_{k-1}| \, |f(\xi_k)|$$

$$\leq \sum_{k=1}^{n} |z_k - z_{k-1}|.$$

Cont..

Now making $n \to \infty$ and using (1), we get

$$\left| \int_{\mathcal{C}} f(z)dz \leq \lim_{n\to\infty} \mathrm{M}\left(|z_1 - z_0| + |z_2 - z_1| + ... + |z_n - z_{n-1}|\right) \right|$$

$$= \mathrm{M} \lim_{n\to\infty} \left(Chordz_1 z_0 + Chordz_2 z_1 + ... + Chordz_n z_{n-1} \right)$$

$$= \mathrm{M} \left(arcz_1 z_0 + arcz_2 z_1 + ...arcz_n z_{n-1} \right)$$

$$= \mathrm{M} \text{ Arc length of Contour } \mathcal{C}$$

$$= \mathrm{M}\mathcal{L}.$$

Thus, $\left|\int_{\mathcal{C}} f(z)dz\right| \leq \mathrm{M}\mathcal{L}$.

EXAMPLE

1.Evaluate $\int_{\mathcal{C}} \frac{1}{z} dz$; where

$$\mathcal{C}: \ x = a\cos\theta, \ y = a\sin\theta, \ \ 0 \leq \theta \leq \pi.$$

Solution $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \left(\frac{x}{x^2+y^2}\right) + i\left(\frac{-y}{x^2+y^2}\right) = \left(\frac{x}{a^2} + i\frac{-y}{a^2}\right)$
Here, $u(x,y) = \frac{x}{a^2}$ and $v(x,y) = \frac{-y}{a^2}$.
Thus

$$I = \int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}} (u+iv)dx + \int_{\mathcal{C}} (-v+iu)dy$$

$$= I_1 + I_2.$$

Cont...

Here,

$$I_1 = \int_a^{-a} \left(\frac{x}{a^2} + i\frac{-y}{a^2} \right) dx$$

$$= \int_a^{-a} \left(\frac{x}{a^2} + i\left(\frac{-1}{a^2} \right) \sqrt{a^2 - x^2} \right) dx$$

$$= 0 + i\left(\frac{-1}{a^2} \right) \int_a^{-a} \sqrt{a^2 - x^2} dx$$

$$= i\frac{\pi}{2}.$$

Similarly, $I_2 = i\frac{\pi}{2}$ hence $I = i\pi$.


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