IMPROPER INTEGRAL

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IMPROPER INTEGRALS

Improper Integral

:TYPE-I
Infinite Limits of Integration

Example

$$\int_{1}^{\infty} \frac{1}{x^2}\, dx$$

:TYPE-II
Discontinuous Integrand
Integrands with Vertical Asymptotes

Example

$$\int_{-1}^{1} \frac{1}{x^2}\, dx$$

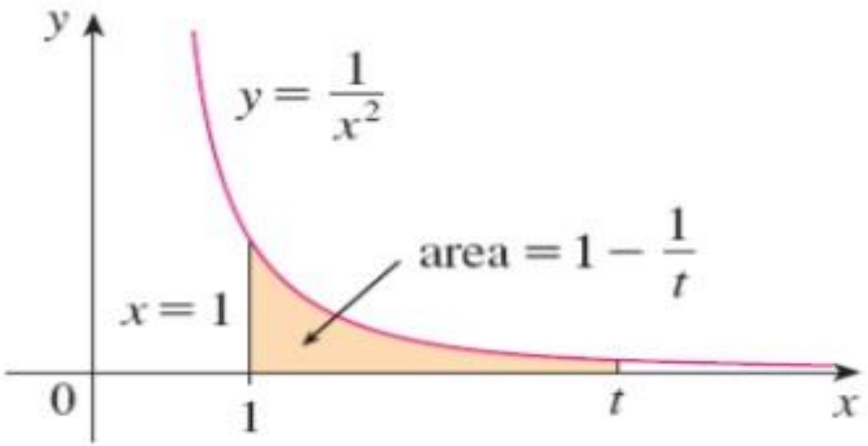
IMPROPER INTEGRALS

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\left(\int_a^b f(x)\,dx\right)$$

Example

$$\int_1^\infty \frac{1}{x^2}\,dx$$



IMPROPER INTEGRALS

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty}\left(\int_{a}^{b} f(x)\,dx \right)$$

Example

$$\int_{-\infty}^{0} xe^{x}\,dx$$

IMPROPER INTEGRALS

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1

$$\int_a^{\infty} f(x)\, dx = \lim_{b \to \infty}\left(\int_a^b f(x)\, dx \right)$$

$$\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty}\left(\int_a^b f(x)\, dx \right)$$

The improper integrals

$$\int_a^{\infty} f(x)\, dx \qquad \int_{-\infty}^a f(x)\, dx$$

are called **convergent** **if the** corresponding limit exists and **divergent** **if the limit does not exist.**

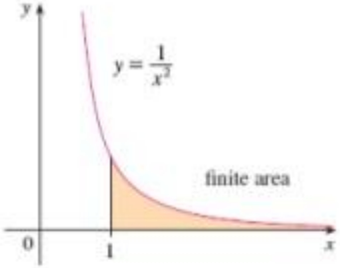
IMPROPER INTEGRALS

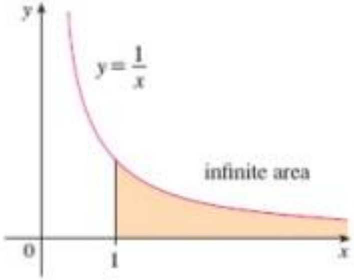
DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\left(\int_a^t f(x)\,dx\right)$$

Example

$$\int_1^{\infty} \frac{1}{x}\,dx$$





IMPROPER INTEGRALS

The improper integral $\int_0^{\pi/2} \frac{\cos x}{1 - \sin x}\, dx$

(a) diverges

(b) converges and has the value 0

(c) converges and has the value $\frac{\pi}{4}$

(d) converges and has the value π

(e) converges and has the value $\frac{\pi}{2}$

IMPROPER INTEGRALS

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 1

$$\int_{-\infty}^{\infty} f(x)\,dx$$ **convergent**

If both improper integrals

$$\int_{a}^{\infty} f(x)\,dx \qquad \int_{-\infty}^{a} f(x)\,dx$$

are convergent

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

Example

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

EXAMPLE 4 For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

convergent?

:Memorize

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

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IMPROPER INTEGRALS

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2

If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

Example

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$$

IMPROPER INTEGRALS

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2

If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

Example

$$\int_0^{\pi/2} \sec x \, dx$$

Example

$$\int_0^1 \frac{1}{1-x}\,dx$$

IMPROPER INTEGRALS

DEFINITION OF AN IMPROPER INTEGRAL OF TYPE 2

If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example

$$\int_0^3 \frac{1}{x-1}\,dx$$

WARNING If we had not noticed the asymptote $x = 1$ in Example 7 and had instead confused the integral with an ordinary integral, then we might have made the following erroneous calculation:

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big]_0^3 = \ln 2 - \ln 1 = \ln 2$$

This is wrong because the integral is improper and must be calculated in terms of limits.

From now on, whenever you meet the symbol $\int_a^b f(x)\,dx$ you must decide, by looking at the function f on $[a, b]$, whether it is an ordinary definite integral or an improper integral.

IMPROPER INTEGRALS

The improper integral $\int_0^3 \frac{3\,dx}{x^2 - 5x + 4}$ is

F092

(a) convergent and its value is 0

(b) convergent and its value is $\ln 4$

(c) convergent and its value is $\ln 3$

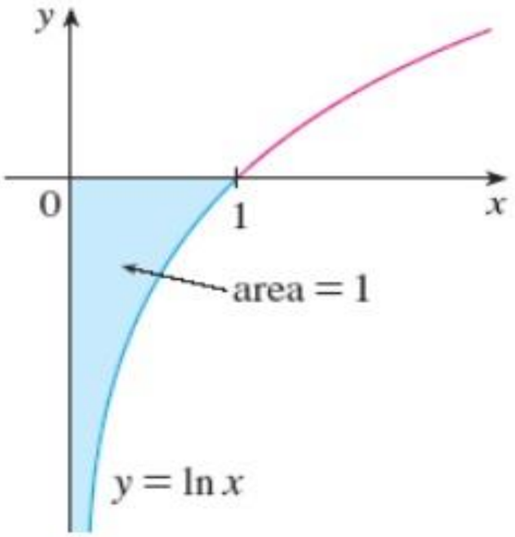
(d) convergent and its value is $\ln 2$

(e) divergent

IMPROPER INTEGRALS

Example

$$\int_0^1 \ln x \, dx$$



IMPROPER INTEGRALS

THEOREM 2—Direct Comparison Test Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges.

2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

EXAMPLE 7

(a) $$\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$$

EXAMPLE 7

(b) $$\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$$

IMPROPER INTEGRALS

THEOREM 3—Limit Comparison Test If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^{\infty} f(x)\,dx \qquad \text{and} \qquad \int_a^{\infty} g(x)\,dx$$

both converge or both diverge.

EXAMPLE 8 Show that

$$\int_1^{\infty} \frac{dx}{1 + x^2}$$

converges

EXAMPLE 9

Investigate the convergence of

$$\int_1^{\infty} \frac{1 - e^{-x}}{x}\,dx.$$

IMPROPER INTEGRALS

Math 102 Final Exam Term 112

If k is a positive real number such that the improper integral $\int_1^\infty \frac{e^{x^k}}{x^{1-k}}\,dx$ is **divergent**, then

(a) $1 < k < 2$ only .

(b) $k > 1$ only

(c) $0 < k < 1$ only

(d) $k > 2$ only

(e) k is any positive real number.

F112

IMPROPER INTEGRALS

The improper integral $\int_0^2 \frac{1}{\sqrt[5]{x-1}}\,dx$

(a) converges and its value is $\frac{1}{4}$

(b) converges and its value is $\frac{5}{4}$

(c) converges and its value is $\frac{5}{2}$

(d) converges and its value is 0

(e) diverges

IMPROPER INTEGRALS

The improper integral $\int_{-\infty}^{1} \frac{1}{2} e^{2x}\, dx$ is

F092

(a) convergent and its value is e

(b) convergent and its value is $e^3/8$

(c) convergent and its value is $e/2$

(d) convergent and its value is $e^2/4$

(e) divergent

IMPROPER INTEGRALS

The improper integral $\int_1^{\infty} \frac{e^{-2\sqrt{x}}}{\sqrt{x}}\, dx$ is

F102

(a) Convergent and its value is 0

(b) Convergent and its value is e^{+2}

(c) Convergent and its value is $-e^{-2}$

(d) Convergent and its value is e^{-2}

(e) Divergent

Evaluating Improper Integrals

Evaluate the integrals in Exercises 1–34 without using tables.

1. $\int_0^\infty \frac{dx}{x^2+1}$

2. $\int_1^\infty \frac{dx}{x^{1.001}}$

3. $\int_0^1 \frac{dx}{\sqrt{x}}$

4. $\int_0^4 \frac{dx}{\sqrt{4-x}}$

5. $\int_{-1}^1 \frac{dx}{x^{2/3}}$

6. $\int_{-8}^1 \frac{dx}{x^{1/3}}$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

8. $\int_0^1 \frac{dr}{r^{0.999}}$

9. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\int_{-\infty}^2 \frac{2\,dx}{x^2+4}$

11. $\int_2^\infty \frac{2}{v^2-v}\,dv$

12. $\int_2^\infty \frac{2\,dt}{t^2-1}$

13. $\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$

14. $\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

18. $\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$

20. $\int_0^\infty \frac{16\tan^{-1} x}{1+x^2}\,dx$

21. $\int_{-\infty}^0 \theta e^\theta\,d\theta$

22. $\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$

Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\int_0^{\pi/2} \tan\theta \, d\theta$

36. $\int_0^{\pi/2} \cot\theta \, d\theta$

37. $\int_0^{\pi} \frac{\sin\theta \, d\theta}{\sqrt{\pi - \theta}}$

38. $\int_{-\pi/2}^{\pi/2} \frac{\cos\theta \, d\theta}{(\pi - 2\theta)^{1/3}}$

39. $\int_0^{\ln 2} x^{-2} e^{-1/x} \, dx$

40. $\int_0^{1} \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx$

41. $\int_0^{\pi} \frac{dt}{\sqrt{t} + \sin t}$

42. $\int_0^{1} \frac{dt}{t - \sin t}$ (*Hint:* $t \geq \sin t$ for $t \geq 0$)

43. $\int_0^{2} \frac{dx}{1 - x^2}$

44. $\int_0^{2} \frac{dx}{1 - x}$

45. $\int_{-1}^{1} \ln |x|\, dx$

46. $\int_{-1}^{1} -x \ln |x|\, dx$

47. $\int_1^{\infty} \frac{dx}{x^3 + 1}$

48. $\int_4^{\infty} \frac{dx}{\sqrt{x} - 1}$

49. $\int_2^{\infty} \frac{dv}{\sqrt{v - 1}}$

50. $\int_0^{\infty} \frac{d\theta}{1 + e^{\theta}}$

51. $\int_0^{\infty} \frac{dx}{\sqrt{x^6 + 1}}$

52. $\int_2^{\infty} \frac{dx}{\sqrt{x^2 - 1}}$

53. $\int_1^{\infty} \frac{\sqrt{x + 1}}{x^2}\, dx$

54. $\int_2^{\infty} \frac{x\, dx}{\sqrt{x^4 - 1}}$

55. $\int_{\pi}^{\infty} \frac{2 + \cos x}{x}\, dx$

56. $\int_{\pi}^{\infty} \frac{1 + \sin x}{x^2}\, dx$

57. $\int_4^{\infty} \frac{2\, dt}{t^{3/2} - 1}$

58. $\int_2^{\infty} \frac{1}{\ln x}\, dx$

59. $\int_1^{\infty} \frac{e^x}{x}\, dx$

60. $\int_{e^e}^{\infty} \ln (\ln x)\, dx$

61. $\int_1^{\infty} \frac{1}{\sqrt{e^x - x}}\, dx$

62. $\int_1^{\infty} \frac{1}{e^x - 2^x}\, dx$

Theory and Examples

65. Find the values of p for which each integral converges.

a. $\displaystyle\int_1^2 \frac{dx}{x(\ln x)^p}$ b. $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$

66. $\int_{-\infty}^{\infty} f(x)\,dx$ **may not equal** $\lim_{b\to\infty} \int_{-b}^{b} f(x)\,dx$ Show that

$$\int_0^\infty \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0.$$

Exercises 67–70 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the x-axis.

67. Find the area of the region.

68. Find the centroid of the region.

69. Find the volume of the solid generated by revolving the region about the y-axis.