Sequence and Series

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12. 1 A sequence is…

(a) an ordered list of objects.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \cdots$$

(b) A function whose domain is a set of integers

Domain: 1, 2, 3, 4, …, *n*…

Range $a_1, a_2, a_3, a_4, \ldots a_n \ldots$

{(1, 1), (2, ½), (3, ¼), (4, 1/8) ….}

Finding patterns

Describe a pattern for each sequence. Write a formula for the nth term

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots \qquad \frac{1}{2^{n-1}}$$

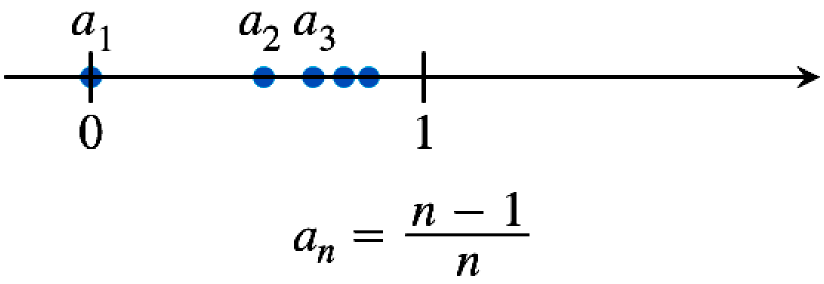
$$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120} \ldots \qquad \frac{1}{n!}$$

$$\frac{1}{4}, \frac{4}{9}, \frac{9}{16}, \frac{16}{25}, \frac{25}{36} \ldots \qquad \frac{n^2}{(n+1)^2}$$

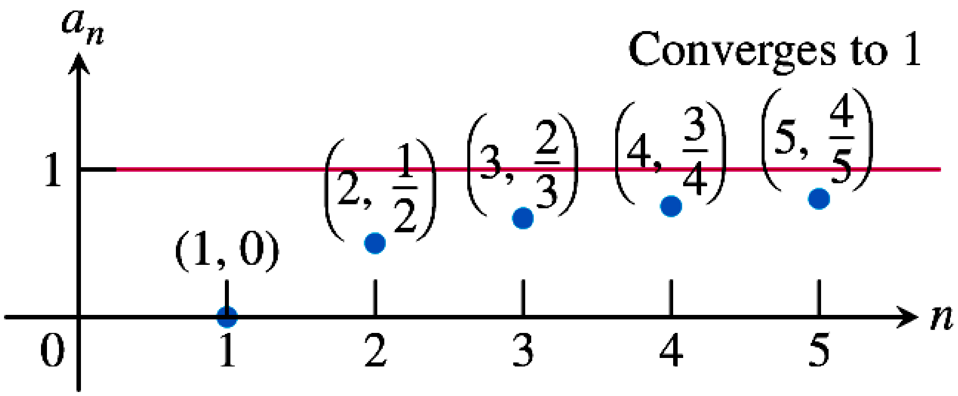
Write the first 5 terms for $a_n = \frac{n-1}{n}$

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \cdots \frac{n-1}{n} \cdots$$

On a number line



As a function

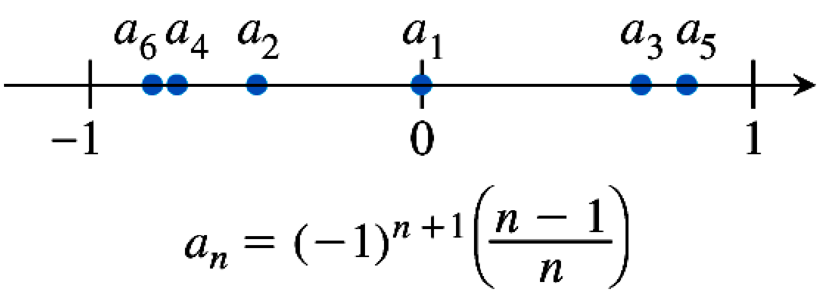


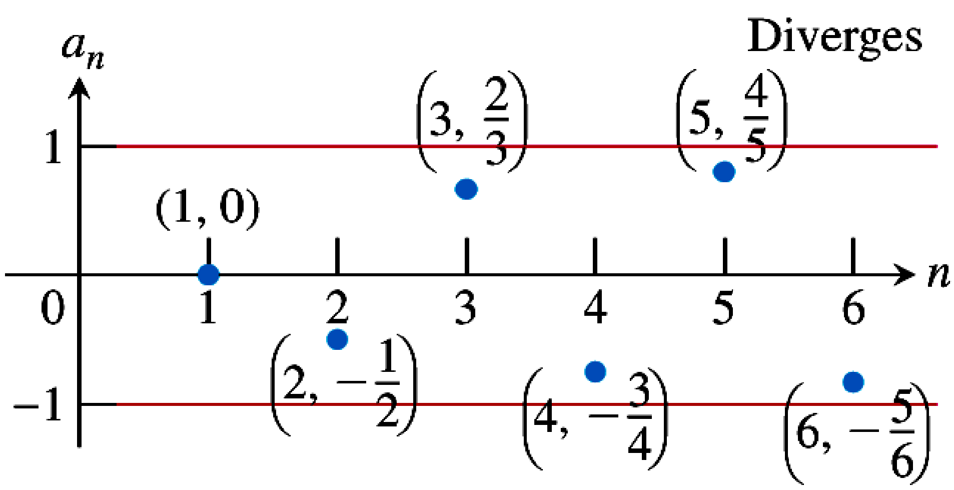
The terms in this sequence get closer and closer to 1. The sequence CONVERGES to 1.

Write the first 5 terms

$$a_n = \frac{(-1)^{n+1}(n-1)}{n}$$

$$0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}. - + - \ldots \frac{n-1}{n} \ldots$$

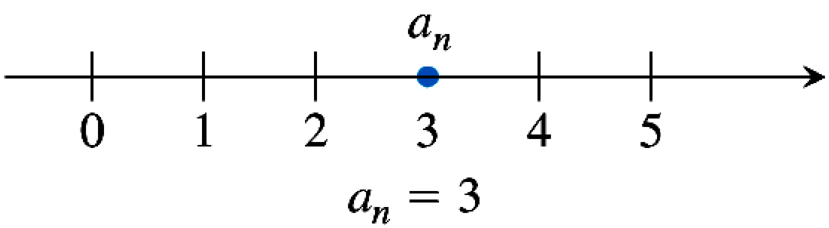


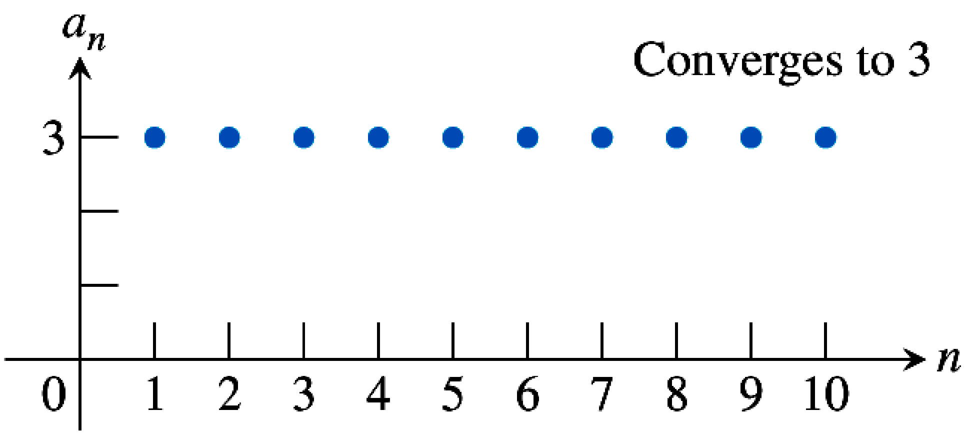


The terms in this sequence do not get close to Any single value. The sequence Diverges

Write the terms for $a_n = 3$

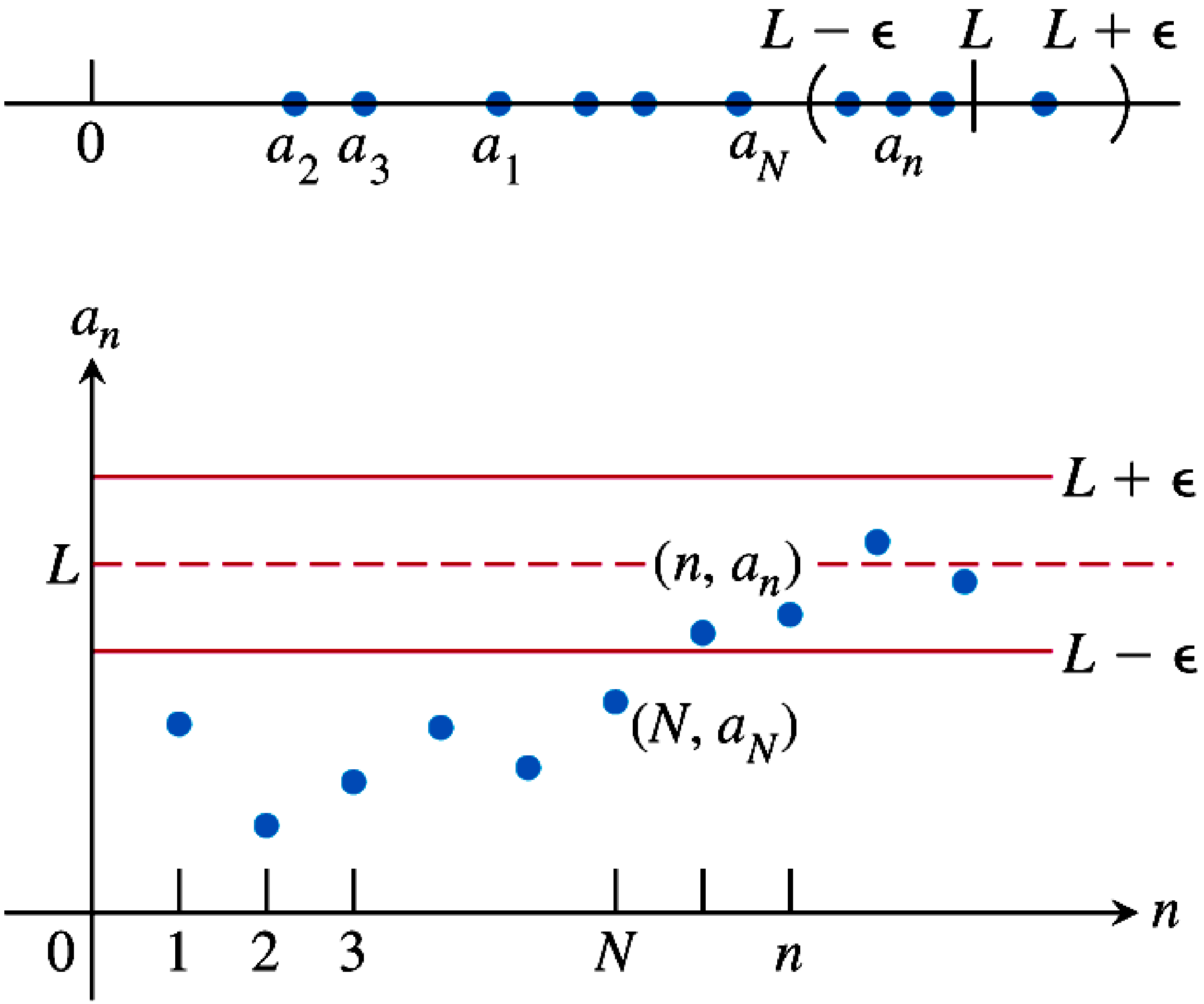
The terms are 3, 3, 3, …3





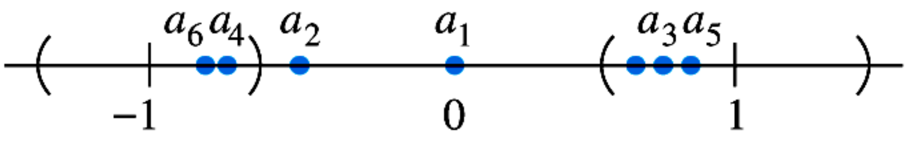
The sequence converges to 3.

y= L is a horizontal asymptote when sequence converges to L.



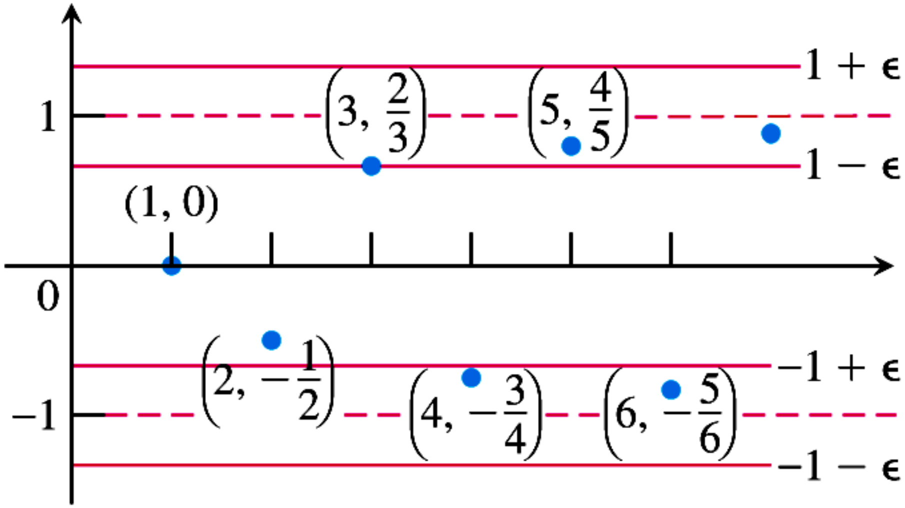
A sequence that diverges

$$a_n = \frac{(-1)^{n+1}(n-1)}{n}$$



$$a_n = (-1)^{n+1}\left(\frac{n-1}{n}\right)$$

Neither the ϵ-interval about 1 nor the ϵ-interval about −1 contains all a_n satisfying $n \geq N$ for some N.



Sequences

Write the first 5 terms of the sequence.
Does the sequence converge? If so, find the value.

$$a_n = \frac{(-1)^{n+1}}{2n-1} \qquad 1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \frac{1}{9} \qquad \lim_{n\to\infty} \frac{(-1)^{n+1}}{2n-1} = 0$$

The sequence converges to 0.

$$a_n = (-1)^{n+1}\left(1-\frac{1}{n}\right) \quad 0, -\frac{1}{2}, +\frac{2}{3}, -\frac{3}{4}, \frac{4}{5} \quad \lim_{n\to\infty} (-1)^{n}\left(1-\frac{1}{n}\right) \textit{ does not exist}$$

The sequence diverges.

12.2 Infinite Series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Represents the sum of the terms in a sequence.

We want to know if the series converges to a single value i.e. there is a finite sum.

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \ldots$$

The series diverges because $s_n = n$. Note that the **Sequence** $\{1\}$ converges.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots$$

Partial sums of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$

$$s_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$s_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{2}{3}$$

$$s_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{3}{4}$$

and

$$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots \frac{1}{n(n+1)} = \frac{n}{n+1}$$

If the sequence of partial sums converges, the series converges

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \ldots \frac{n}{n+1} \ldots$ Converges to 1 so **series** converges.

Finding sums $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

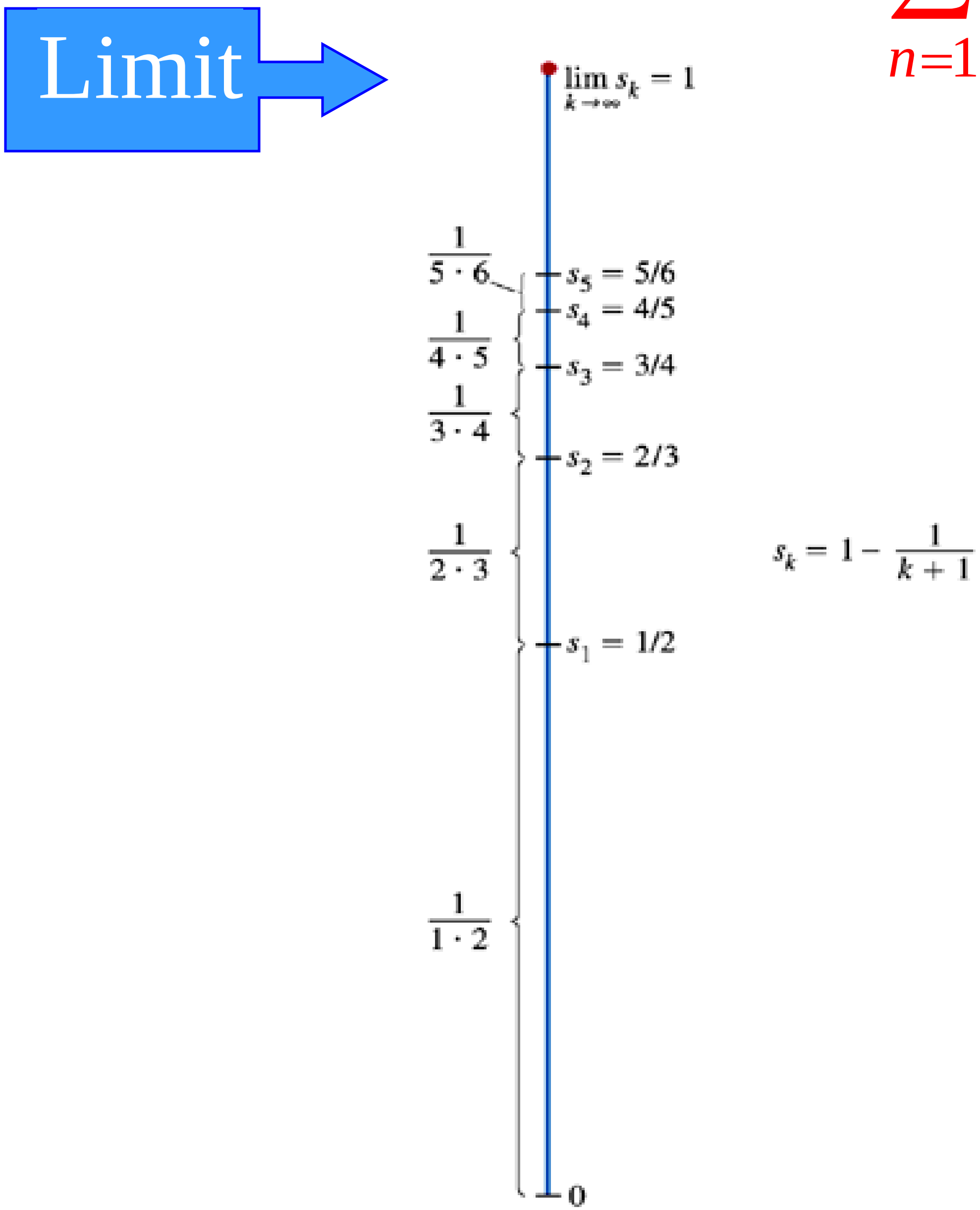
Can use partial fractions to rewrite

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} =$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots \frac{1}{n} - \frac{1}{n+1} + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \to \infty} (1 - \frac{1}{n+1}) = 1$$

The partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$



Geometric Series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Each term is obtained from the preceding number by multiplying by the same number *r*.

Find r (the common ratio)

$$\frac{1}{5} - \frac{1}{25} + \frac{1}{125} - \frac{1}{625} + \ldots$$

$$\frac{2}{3} + \frac{4}{3} + \frac{8}{3} + \frac{16}{3} + \ldots$$

$$\sum_{n=1}^{\infty} ar^{n-1}$$

Is a Geometric Series

Where a = first term and r=common ratio

Write using series notation

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots \qquad \sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

$$\frac{3}{5}-\frac{12}{25}+\frac{48}{125}-\frac{192}{625}+\ldots \qquad \sum_{n=1}^{\infty}\frac{3}{5}\left(-\frac{4}{5}\right)^{n-1}$$

$$\frac{2}{3}+\frac{4}{3}+\frac{8}{3}+\frac{16}{3}+\ldots \qquad \sum_{n=1}^{\infty}\frac{2}{3}(2)^{n-1}$$

The sum of a geometric series

$$s_n = a + ar + ar^2 + ar^3 + \dots ar^{n-1}$$ Sum of n terms

$$rs_n = ar + ar^2 + ar^3 + \dots ar^n$$ Multiply each term by r

$$s_n - rs_n = a - ar^n$$ subtract

$$s_n = \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r}, r \neq 1$$

$$if\ |r| < 1, \qquad r^n \to 0\ as\ n \to \infty$$

Geometric series converges to $s_n = \frac{a}{1-r}, |r| < 1$

If r>1 the geometric series diverges.

Find the sum of a Geometric Series

Where a = first term and r=common ratio

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r|<1$$

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

$$\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$\sum_{n=1}^{\infty} \frac{3}{5}\left(-\frac{4}{5}\right)^{n-1}$$

$$\frac{\frac{3}{5}}{1+\frac{4}{5}} = \frac{3}{9} = \frac{1}{3}$$

$$\sum_{n=1}^{\infty} \frac{2}{3}(2)^{n-1}$$

The series diverges.

Repeating decimals-Geometric Series

$$0.0808\overline{08} = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \ldots$$

$$a = \frac{8}{10^2} \quad and \quad r = \frac{1}{10^2}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \frac{8}{10^2}\left(\frac{1}{10^2}\right)^{n-1}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{\frac{8}{10^2}}{1-\frac{1}{10^2}} = \frac{8}{99}$$

The repeating decimal is equivalent to 8/99.

Series known to converge or diverge

1. A geometric series with $| r | < 1$ converges

2. A repeating decimal converges

3. Telescoping series converge

A necessary condition for convergence:
Limit as *n* goes to infinity for nth term in sequence is 0.

nth term test for divergence:
If the limit as n goes to infinity for the nth term is not 0, the series DIVERGES!

Convergence or Divergence?

$$\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty} (1.075)^n$$

$$\sum_{n=1}^{\infty} \frac{4}{2^n}$$

A sequence in which each term is less than or equal to the one before it is called a **monotonic non-increasing** sequence. If each term is greater than or equal to the one before it, it is called **monotonic non-decreasing.**

A monotonic <u>sequence</u> that is bounded
Is convergent.

A <u>series</u> of non-negative terms converges
If its partial sums are bounded from above.

12. 3 The Integral Test

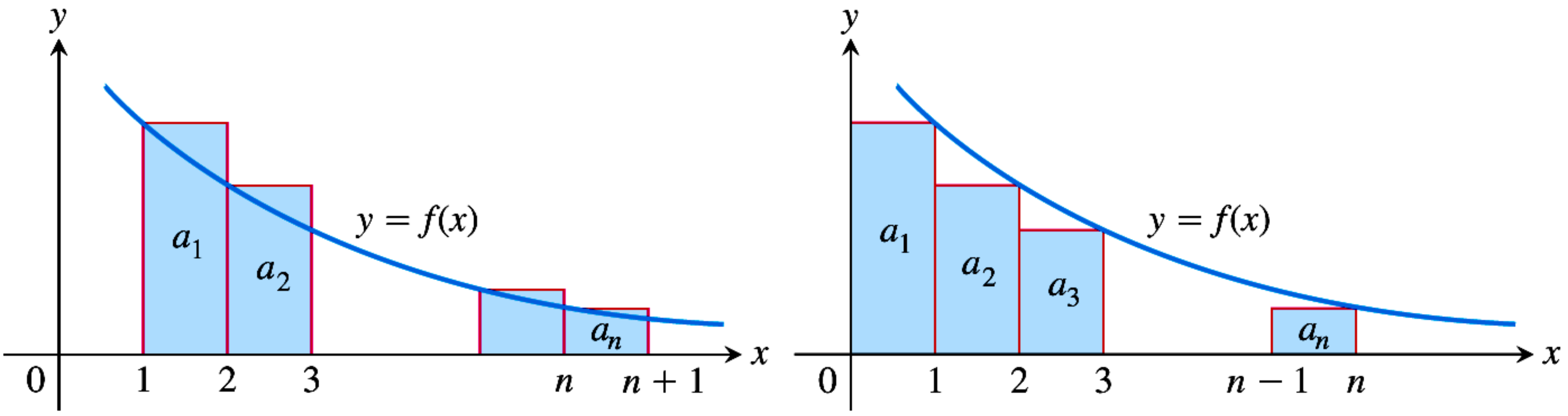
Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$ where f is a continuous positive, decreasing function of x for all $x \geq N$. Then the series and the corresponding integral shown **both converge** of **both diverge**.

$f(n)$ → $$\sum_{n=N}^{\infty} a_n$$

$f(x)$ → $$\int_{N}^{\infty} f(x)\,dx$$

The series and the integral both converge or both diverge

Area in rectangle corresponds to term in sequence



(a)

(b)

Exact area under curve is between

If area under curve is finite, so is area in rectangles

If area under curve is infinite, so is area in rectangles

Using the Integral test

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \qquad a_n = f(n) = \frac{n}{n^2+1} \qquad f(x) = \frac{x}{x^2+1}$$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_1^{b} \frac{2x}{x^2+1}\,dx = \lim_{b\to\infty}\left[\ln(x^2+1)\right]_1^b$$

$$\lim_{b\to\infty}(\ln(b^2+1) - \ln 2) = \infty$$

The improper integral diverges

Thus the series diverges

Using the Integral test

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \qquad a_n = f(n) = \frac{1}{n^2+1} \qquad f(x) = \frac{1}{x^2+1}$$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \left[\arctan x\right]_1^b$$

$$\lim_{b\to\infty} (\arctan b - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

The improper integral converges

Thus the series converges

Harmonic series and p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ Is called a p-series

A p-series converges if p > 1 and diverges
If p < 1 or p = 1.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots.\frac{1}{n} + \ldots.$$

Is called the harmonic series and it
diverges since p =1.

Identify which series converge and which diverge.

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{100}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{\pi}{3}}}$$

$$\sum_{n=1}^{\infty} \frac{3}{5}\left(-\frac{4}{5}\right)^{n-1}$$

12. 4Direct Comparison test

Let $\sum_{n=1}^{\infty} a_n$ be a series with no negative terms

$\sum_{n=1}^{\infty} a_n$ Converges if there is a series $\sum_{n=1}^{\infty} c_n$

Where the terms of a_n are less than or equal to the terms of c_n for all n>N.

$\sum_{n=1}^{\infty} a_n$ Diverges if there is a series $\sum_{n=1}^{\infty} d_n$

Where the terms of a_n are greater than or equal to the terms of d_n for all n>N.

Limit Comparison test

$$\lim_{x\to\infty} \frac{a_n}{b_n} = c, \qquad 0 < c < \infty$$

Then the following series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

both converge or both diverge:

$\lim_{x\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ Converges then $\sum_{n=1}^{\infty} a_n$ Converges

$\lim_{x\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ Diverges then $\sum_{n=1}^{\infty} a_n$ Diverges

Convergence or divergence?

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

Alternating Series

A series in which terms alternate in sign

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2^n} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$$

$$\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \ldots$$

Alternating Series Test

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \ldots$$

Converges if:

✓ a_n is always positive

✓ $a_n \geq a_{n+1}$ for all $n \geq N$ for some integer N.

✓ $a_n \to 0$

If any one of the conditions is not met, the Series diverges.

Absolute and Conditional Convergence

- A series $\sum_{n=N}^{\infty} a_n$ is **absolutely convergent** if the corresponding series of absolute values $\sum_{n=N}^{\infty} |a_n|$ converges.

- A series that converges but does not converge absolutely, converges conditionally.

- Every absolutely convergent series converges. (Converse is false!!!)

Is the given series convergent or divergent? If it is convergent, its it absolutely convergent or conditionally convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

a) Is the given series convergent or divergent? If it is convergent, its it absolutely convergent or conditionally convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n} = -\frac{1}{3} - \frac{1}{9} + \frac{1}{27} + \frac{1}{81} - \dots$$

This is not an alternating series, but since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n(n+1)/2}}{3^n} \right| = \sum_{n=1}^{\infty} \frac{1}{3^n}$$

Is a convergent geometric series, then the given Series is <u>absolutely convergent</u>.

b) Is the given series convergent or divergent? If it is convergent, its it absolutely convergent or conditionally convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)} = -\frac{1}{\ln 2} + \frac{1}{\ln 3} - \frac{1}{\ln 4} + \dots\dots$$

Converges by the Alternating series test.

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\ln(n+1)} \right| = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \dots\dots$$

Diverges with direct comparison with the harmonic Series. The given series is <u>conditionally convergent</u>.

c) Is the given series convergent or divergent? If it is convergent, its it absolutely convergent or conditionally convergent?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} +$$

By the nth term test for divergence, the series Diverges.

d) Is the given series convergent or divergent? If it is convergent, its it absolutely convergent or conditionally convergent?

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} = -\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}$$

Converges by the alternating series test.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}$$

Diverges since it is a p-series with $p < 1$. The Given series is conditionally convergent.

The Ratio Test

Let $\sum_{n=N}^{\infty} a_n$ be a series with positive terms and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

- The series converges if $\rho < 1$
- The series diverges if $\rho > 1$
- The test is inconclusive if $\rho = 1$.

The Root Test

Let $\sum_{n=N}^{\infty} a_n$ be a series with non-zero terms and

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

Then

- The series converges if L< 1
- The series diverges if L > 1 or is infinite
- The test is inconclusive if L= 1.

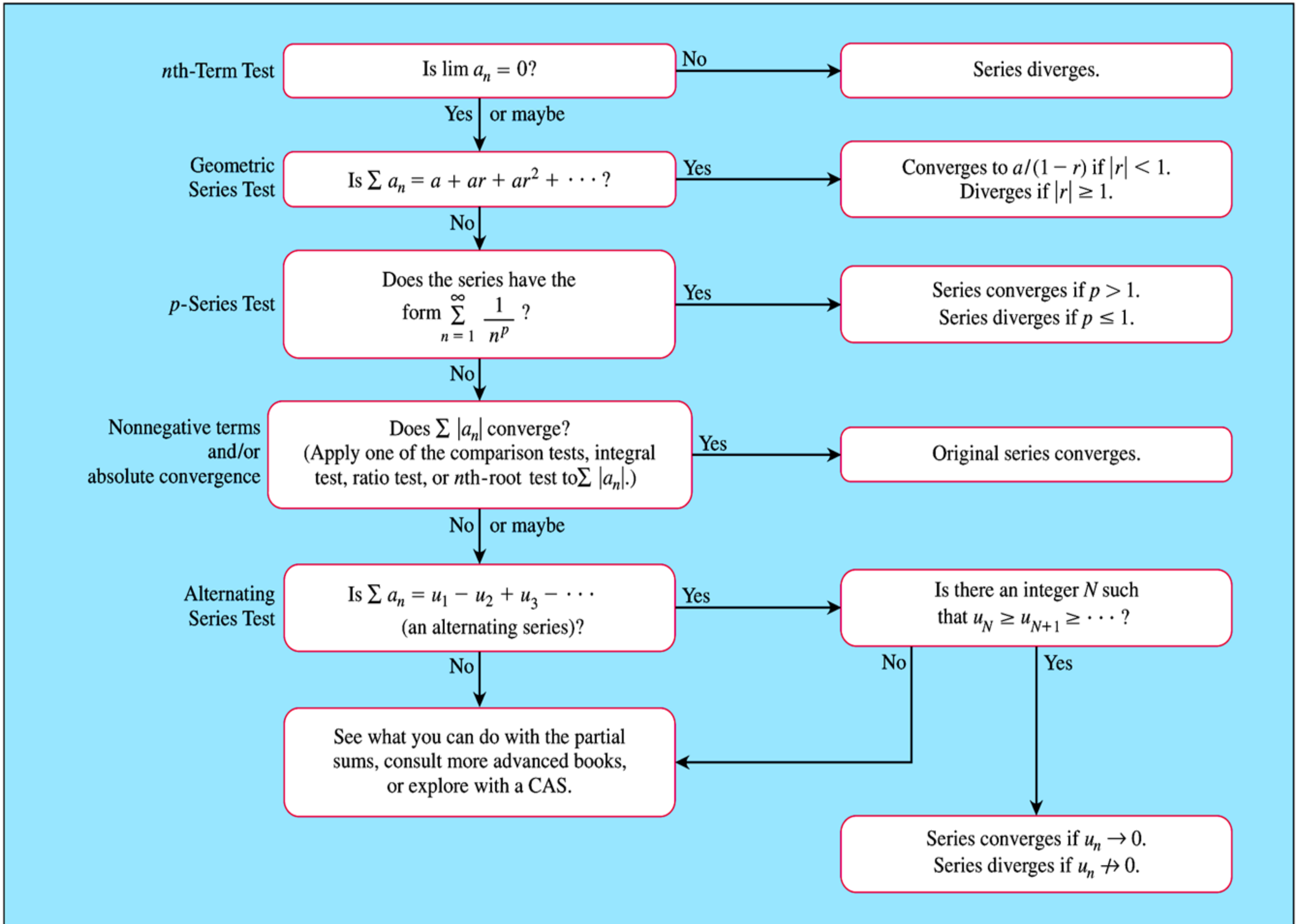
Convergence or divergence?

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^{n+1}}$$

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$$

Procedure for determining Convergence



THANKS YOU